Algebraic Analysis of Data Fusion with Ellipsoidal Intersection

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***Abstract*— For decentralized fusion problems, ellipsoidal intersection has been proposed as an efficient fusion rule that provides less conservative results as compared to the well-know covariance intersection method. Ellipsoidal intersection relies on the computation of a common estimate that is shared by the estimates to be fused. In this paper, an algebraic reformulation of ellipsoidal intersection is discussed that circumvents the computation of the common estimate. It is shown that ellipsoidal intersection corresponds to an internal ellipsoidal approximation of the intersection of covariance ellipsoids. An interesting result is that ellipsoidal intersection can be computed with the aid of the Bar-Shalom/Campo fusion formulae. This is achieved by assuming a specific correlation structure between the estimates to be fused.**

I. INTRODUCTION

State estimation methods, of which the Kalman filter [1] is one of the most prominent procedures, are utilized to infer information about a system's state. The Kalman filter recursively computes a state estimate based on prior information, a process model, and measurements acquired from sensor devices. In typical network-based sensor systems [2], it is often not a single instance that computes an estimate but a multiplicity of nodes, each of which is equipped with its own state estimation system. Such distributed estimation strategies eliminate the need to send all sensor data to a center node where a single Kalman filter computes an estimate. Instead, sensor data can be processed locally. With nodes running local Kalman filters, fusion methods then provide the means to combine estimates from different nodes. In a centralized approach, it is at least the data sink that has to apply a fusion algorithm to the incoming estimates in order to form a global estimation result. In fully decentralized estimation networks, nodes can benefit from an exchange of each other's estimates by applying fusion methods locally.

Compared to a centralized processing of all acquired measurements, distributed and decentralized processing schemes often require more elaborate algorithms that take care of possible correlations between the local estimation errors [3]. Although distributed implementations [4], [5] of the Kalman filter algorithm can be established, they are often inapplicable because of being highly susceptible to node failures and changes in the network topology as discussed in [6], [7]. Furthermore, such distributed schemes still require a central data sink where a fused estimate is computed. Compared with this, fully decentralized estimation systems do not rely on a dedicated fusion center, and they often consist of sensor nodes that are each equipped with a local Kalman filter. Each node independently computes an estimate, which is optimal given the locally acquired measurements. By exchanging and fusing estimates, the locally attainable estimation quality can be further improved. Although fully-decentralized processing strategies are attractive in many applications, two important aspects have to be addressed. First, an optimal and consistent fusion of estimates has to exploit the underlying correlation structure, which hence must be known. The optimal result is then given by the Bar-Shalom/Campo fusion rule [8] for the two-sensor fusion problem or its generalization [9] for the multi-sensor case. Second, the optimal fusion of all local estimates does not provide the optimal estimate given all available measurements with respect to the mean-squared error [10], i.e., local Kalman filters that exchange and fuse their estimates do not reach the same estimation quality as a single Kalman filter that has access to all sensors. This discrepancy can be addressed by tracklet-based fusion methods [11], [12] or joint state representations [13], when certain constraints on the transmission policy are met.

The major problem when estimates are to be fused is the correct treatment of possible correlations. Although an optimal reconstruction of correlations is possible for special network topologies [14], bookkeeping of the correlation structure is often too cumbersome or even impossible. For this reason, conservative fusion methods like *covariance intersection* (CI) [15]–[17] guarantee consistent fusion results irrespective of the underlying correlation structure. Although these methods are suboptimal compared with the *Bar-Shalom/Campo* (B/C) fusion rule, [18] have shown that CI tightly bounds the entirety of possible error covariance matrices. However, CI often turns out to be too conservative as strong correlations seldom occur. Starting from this observation, [19] has proposed to consider the largest internal ellipsoid in order to compute the fused error covariance matrix, which is smaller than the covariance matrix reported by CI. This approach has been revisited in [20] by refining the computation of the fused estimate. A recent approach gaining considerable attention is *ellipsoidal intersection* (EI) [21]. Its effectiveness has been demonstrated in several case studies [22] and comparisons [23] with other approaches. In this work, the derivation of EI is revisited, and an in-depth analysis is carried out in order to provide additional insights, and its relationship to the internal ellipsoid approximation and B/C fusion rule is discussed.

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NOTATIONS

An underlined variable $\underline{x} \in \mathbb{R}^n$ denotes a real-valued vector, and lowercase boldface letters $\underline{\mathbf{x}}$ are used for random quantities. Matrices are written in uppercase boldface letters $\mathbf{C} \in \mathbb{R}^{n\times n}$, and $\mathbf{C}^{-1}$ and $\mathbf{C}^{\mathrm{T}}$ are its inverse and transpose, respectively. For the ith element in $\underline{x}$ and the ith diagonal element in $\mathbf{C}$, the notations $(\underline{x})_i$ and $(\mathbf{C})_{ii}$ are used. $\mathbf{C} \geq \mathbf{C}'$ implies that the difference $\mathbf{C} - \mathbf{C}'$ is positive semi-definite. The notation $(\hat{\underline{\mathbf{x}}}, \mathbf{C})$ is used for an estimate $\hat{\underline{\mathbf{x}}}$ of $\underline{\mathbf{x}}$ that has the error covariance matrix $\mathbf{C} = \mathrm{E}[\tilde{\underline{\mathbf{x}}}\,\tilde{\underline{\mathbf{x}}}^{\mathrm{T}}]$ with $\tilde{\underline{\mathbf{x}}} = \hat{\underline{\mathbf{x}}} - \underline{\mathbf{x}}$. $\mathcal{E}(\hat{\underline{c}}, \mathbf{X})$ is an ellipsoid with center $\hat{\underline{c}}$ and shape matrix $\mathbf{X}$.

II. CONSIDERED FUSION PROBLEM

For the derivation of EI, a special decomposition of the estimates to be fused is employed [21]. Two consistent estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$ are considered, and it is assumed that both estimates can be represented by the three mutually uncorrelated estimates $(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}}, \boldsymbol{\Lambda}_{\mathsf{A}})$, $(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}}, \boldsymbol{\Lambda}_{\mathsf{B}})$, and $(\hat{\underline{\boldsymbol{\gamma}}}, \boldsymbol{\Gamma})$ according to

$$\hat{\underline{\mathbf{x}}}_{\mathsf{A}} = \mathbf{C}_{\mathsf{A}}\big(\boldsymbol{\Lambda}_{\mathsf{A}}^{-1}\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} + \boldsymbol{\Gamma}^{-1}\hat{\underline{\boldsymbol{\gamma}}}\big) \tag{1a}$$

$$\mathbf{C}_{\mathsf{A}} = \big(\boldsymbol{\Lambda}_{\mathsf{A}}^{-1} + \boldsymbol{\Gamma}^{-1}\big)^{-1} \tag{1b}$$

and

$$\hat{\underline{\mathbf{x}}}_{\mathsf{B}} = \mathbf{C}_{\mathsf{B}}\big(\boldsymbol{\Lambda}_{\mathsf{B}}^{-1}\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}} + \boldsymbol{\Gamma}^{-1}\hat{\underline{\boldsymbol{\gamma}}}\big) \tag{2a}$$

$$\mathbf{C}_{\mathsf{B}} = \big(\boldsymbol{\Lambda}_{\mathsf{B}}^{-1} + \boldsymbol{\Gamma}^{-1}\big)^{-1} \ . \tag{2b}$$

Hence, both estimates share the common estimate $(\hat{\underline{\boldsymbol{\gamma}}}, \boldsymbol{\Gamma})$. The corresponding estimation errors become

$$\begin{aligned}\tilde{\underline{\mathbf{x}}}_{\mathsf{A}} = \hat{\underline{\mathbf{x}}}_{\mathsf{A}} - \underline{\mathbf{x}} &= \mathbf{C}_{\mathsf{A}}\left(\boldsymbol{\Lambda}_{\mathsf{A}}^{-1}\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} + \boldsymbol{\Gamma}^{-1}\hat{\underline{\boldsymbol{\gamma}}}\right) - \underline{\mathbf{x}} \\ &= \mathbf{C}_{\mathsf{A}}\left(\boldsymbol{\Lambda}_{\mathsf{A}}^{-1}(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} - \underline{\mathbf{x}}) + \boldsymbol{\Gamma}^{-1}(\hat{\underline{\boldsymbol{\gamma}}} - \underline{\mathbf{x}})\right) \\ &= \mathbf{C}_{\mathsf{A}}\left(\boldsymbol{\Lambda}_{\mathsf{A}}^{-1}\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} + \boldsymbol{\Gamma}^{-1}\tilde{\underline{\boldsymbol{\gamma}}}\right)\end{aligned}$$

and

$$\tilde{\underline{\mathbf{x}}}_{\mathsf{B}} = \mathbf{C}_{\mathsf{B}}\left(\boldsymbol{\Lambda}_{\mathsf{B}}^{-1}\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}} + \boldsymbol{\Gamma}^{-1}\tilde{\underline{\boldsymbol{\gamma}}}\right) \ ,$$

which leads to the error cross-covariance matrix

$$\mathbf{C}_{\mathsf{AB}} = \mathbf{C}_{\mathsf{BA}}^{\mathrm{T}} = \mathrm{E}\left[\tilde{\underline{\mathbf{x}}}_{\mathsf{A}}\tilde{\underline{\mathbf{x}}}_{\mathsf{B}}^{\mathrm{T}}\right] = \mathbf{C}_{\mathsf{A}}\boldsymbol{\Gamma}^{-1}\mathbf{C}_{\mathsf{B}} \tag{4}$$

since $\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}}$, $\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}}$, and $\tilde{\underline{\boldsymbol{\gamma}}}$ have been assumed to be mutually uncorrelated, i.e., $\mathrm{E}[\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}}\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}}^{\mathrm{T}}] = \mathrm{E}[\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}}\tilde{\underline{\boldsymbol{\gamma}}}^{\mathrm{T}}] = \mathrm{E}[\tilde{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}}\tilde{\underline{\boldsymbol{\gamma}}}^{\mathrm{T}}] = \mathbf{0}$.

A. Optimal Fusion With Known Common Information

When the cross-covariance matrix $\mathbf{C}_{\mathsf{AB}}$ is known and can be exploited, the B/C fusion formulae [8] encompass the optimal linear combination of the estimates with respect to minimizing the mean-squared error. The estimates are fused according to

$$\hat{\underline{\mathbf{x}}}_{\mathrm{B/C}} = \mathbf{K}_{\mathrm{B/C}}\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}} + \mathbf{L}_{\mathrm{B/C}}\,\hat{\underline{\mathbf{x}}}_{\mathsf{B}} \tag{5a}$$

and

$$\begin{aligned}\mathbf{C}_{\mathrm{B/C}} &= \mathbf{K}_{\mathrm{B/C}}\mathbf{C}_{\mathsf{A}}\mathbf{K}_{\mathrm{B/C}}^{\mathrm{T}} + \mathbf{L}_{\mathrm{B/C}}\mathbf{C}_{\mathsf{B}}\mathbf{L}_{\mathrm{B/C}}^{\mathrm{T}} + \\ &\quad \mathbf{K}_{\mathrm{B/C}}\mathbf{C}_{\mathsf{AB}}\mathbf{L}_{\mathrm{B/C}}^{\mathrm{T}} + \mathbf{L}_{\mathrm{B/C}}\mathbf{C}_{\mathsf{BA}}\mathbf{K}_{\mathrm{B/C}}^{\mathrm{T}} \\ &= \mathbf{C}_{\mathsf{A}} - \mathbf{L}_{\mathrm{B/C}}\left(\mathbf{C}_{\mathsf{A}} - \mathbf{C}_{\mathsf{AB}}\right)^{\mathrm{T}}\end{aligned} \tag{5b}$$

with the gains $\mathbf{K}_{\mathrm{B/C}}$ and $\mathbf{L}_{\mathrm{B/C}} = \mathbf{I} - \mathbf{K}_{\mathrm{B/C}}$ given by

$$\mathbf{K}_{\mathrm{B/C}} = \big(\mathbf{C}_{\mathsf{B}} - \mathbf{C}_{\mathsf{BA}}\big)\cdot\big(\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}} - \mathbf{C}_{\mathsf{AB}} - \mathbf{C}_{\mathsf{BA}}\big)^{-1}, \tag{6a}$$

$$\mathbf{L}_{\mathrm{B/C}} = \big(\mathbf{C}_{\mathsf{A}} - \mathbf{C}_{\mathsf{AB}}\big)\cdot\big(\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}} - \mathbf{C}_{\mathsf{AB}} - \mathbf{C}_{\mathsf{BA}}\big)^{-1}. \tag{6b}$$

The gains are determined such that the trace of (5b), which corresponds to the mean-squared error, is minimized. It is important to notice that both (1) and (2) constitute the optimal fusion results of uncorrelated estimates:

- In (1), $(\hat{\underline{\boldsymbol{\gamma}}}, \boldsymbol{\Gamma})$ has been fused with $(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}}, \boldsymbol{\Lambda}_{\mathsf{A}})$.
- In (2), $(\hat{\underline{\boldsymbol{\gamma}}}, \boldsymbol{\Gamma})$ has been fused with $(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}}, \boldsymbol{\Lambda}_{\mathsf{B}})$.

More precisely, (1) and (2) correspond to the important special case of (5) where the cross-covariance matrix is zero and estimates are uncorrelated.

Although formulae (5) represent the optimal linear combination of estimates, they do not necessarily provide the optimal estimate given the available information [10]. Interestingly, this discrepancy can also be seen for the special decompositions (1) and (2). Due to these decompositions, the optimal fusion of the estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$ corresponds to the optimal fusion[1] of the three partial estimates $(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}}, \boldsymbol{\Lambda}_{\mathsf{A}})$, $(\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}}, \boldsymbol{\Lambda}_{\mathsf{B}})$, and $(\hat{\underline{\boldsymbol{\gamma}}}, \boldsymbol{\Gamma})$, i.e.,

$$\hat{\underline{\mathbf{x}}}_{\mathrm{opt}} = \mathbf{C}_{\mathrm{opt}}\big(\boldsymbol{\Lambda}_{\mathsf{A}}^{-1}\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} + \boldsymbol{\Lambda}_{\mathsf{B}}^{-1}\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}} + \boldsymbol{\Gamma}^{-1}\hat{\underline{\boldsymbol{\gamma}}}\big) \tag{7a}$$

and

$$\mathbf{C}_{\mathrm{opt}} = \big(\boldsymbol{\Lambda}_{\mathsf{A}}^{-1} + \boldsymbol{\Lambda}_{\mathsf{B}}^{-1} + \boldsymbol{\Gamma}^{-1}\big)^{-1} \ . \tag{7b}$$

This fusion rule is the result of determining the optimal gains for the combination $\mathbf{K}_1\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{A}} + \mathbf{K}_2\hat{\underline{\boldsymbol{\lambda}}}_{\mathsf{B}} + \mathbf{K}_3\hat{\underline{\boldsymbol{\gamma}}}$ such that the mean-squared error is minimized. Again, it has been exploited that the partial estimates have mutually uncorrelated errors. By considering the corresponding covariance ellipses, the example in Fig. 1(a) immediately unveils that the fusion result (7) differs from (5). The fusion strategy (7) even reports a lower error covariance matrix, i.e., $\mathbf{C}_{\mathrm{opt}} \leq \mathbf{C}_{\mathrm{B/C}}$, as illustrated in Fig. 1(b). Consequently, if a decomposition according to (1) and (2) is known, it should be exploited by means of (7) instead of (5). This observation is underpinned by the following theorem.

Theorem 1: *The fusion rule* (7) *always provides a smaller error than the* B/C *fusion rule* (5). *More precisely, the inequality*

$$\mathbf{C}_{\mathrm{B/C}} \geq \mathbf{C}_{\mathrm{opt}} \tag{8}$$

holds.

Proof: The proof is presented in Appendix A. □

The fusion rule (7) can also be reorganized into

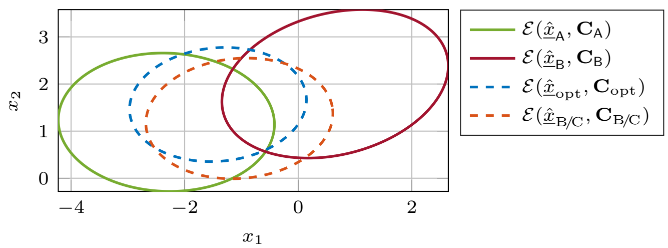
$$\hat{\underline{\mathbf{x}}}_{\mathrm{opt}} = \mathbf{C}_{\mathrm{opt}}\big(\mathbf{C}_{\mathsf{A}}^{-1}\hat{\underline{\mathbf{x}}}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}}^{-1}\hat{\underline{\mathbf{x}}}_{\mathsf{B}} - \boldsymbol{\Gamma}^{-1}\hat{\underline{\boldsymbol{\gamma}}}\big) \tag{9a}$$

and

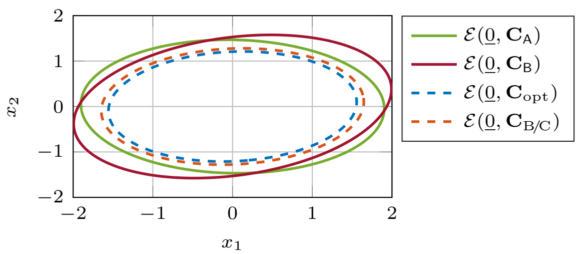
$$\mathbf{C}_{\mathrm{opt}} = \big(\mathbf{C}_{\mathsf{A}}^{-1} + \mathbf{C}_{\mathsf{B}}^{-1} - \boldsymbol{\Gamma}^{-1}\big)^{-1} \ . \tag{9b}$$

This reformulation implies that the estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$ can fused as if they are uncorrelated ($\mathbf{C}_{\mathsf{AB}} = \mathbf{0}$)

[1]The formulae (1), (2), and (7) correspond to the information form [24] of the Kalman filter and can be computed recursively.



(a) Fusion results of two estimates $(\hat{\underline{\mathbf{x}}}_\mathsf{A}, \mathbf{C}_\mathsf{A})$ and $(\hat{\underline{\mathbf{x}}}_\mathsf{B}, \mathbf{C}_\mathsf{B})$ with decompositions (1) and (2).



(b) Fusion results of Fig. (a) centered at the origin.

Fig. 1. Comparison of fusion methods when the decompositions (1) and (2) can be exploited.

and the common estimate $(\hat{\underline{\gamma}}, \mathbf{\Gamma})$ can then be subtracted after fusion. This strategy is employed in the channel filter [25] and its nonlinear counterpart [26], where common information is kept track of. It is important to emphasize that (7) can only be applied if a common estimate $(\hat{\underline{\gamma}}, \mathbf{\Gamma})$ exists and can be exploited.

B. Suboptimal Fusion With Unknown Common Information

An often encountered problem is the treatment of an unknown correlation structure when estimates are to be fused. The most general solution is provided by CI [15], which yields consistent fusion results irrespective of the unknown cross-covariance matrix $\mathbf{C}_\mathsf{AB}$. For this purpose, CI replaces (5) by a conservative formulation, and in particular, (5b) is replaced by an upper bound for all possible $\mathbf{C}_\mathsf{AB}$. It has been shown in [18] that CI is the optimal approach when the cross-covariance matrix $\mathbf{C}_\mathsf{AB}$ is unknown.

The EI algorithm has been proposed as a conservative fusion rule for state estimates given by (1) and (2) when the common information $(\hat{\underline{\gamma}}, \mathbf{\Gamma})$ is unknown to the fusion system. By design, EI is tailored to the specific correlation structure in (4). A simple example unveils that correlations between two estimates may display a different structure than (4).

Example 2: *Possible covariance matrices are* $\mathbf{C}_\mathsf{A} = \left[\begin{smallmatrix} 4 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\mathbf{C}_\mathsf{B} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right]$ *with cross-covariance matrix* $\mathbf{C}_\mathsf{AB} = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. *According to* (4), *the common term would yield* $\mathbf{\Gamma}^{-1} = \mathbf{C}_\mathsf{A}^{-1}\mathbf{C}_\mathsf{AB}\mathbf{C}_\mathsf{B}^{-1} = \left[\begin{smallmatrix} 0.5 & 0 \\ 0 & 0.5 \end{smallmatrix}\right]$. *Apparently, the differences* $\mathbf{\Lambda}_\mathsf{A}^{-1} = \mathbf{C}_\mathsf{A}^{-1} - \mathbf{\Gamma}^{-1}$ *and* $\mathbf{\Lambda}_\mathsf{B}^{-1} = \mathbf{C}_\mathsf{B}^{-1} - \mathbf{\Gamma}^{-1}$ *are not positive semi-definite, which is in conflict with* (1b) *and* (2b).

As a consequence, EI does not constitute a general conservative fusion rule as CI does. However, it is often desirable to employ fusion methods that are less conservative than CI. Keeping Theorem 1 in mind, we can expect that exploiting the decompositions (1) and (2) for fusion yields better results even if conservative approximations of the parameters $(\hat{\underline{\gamma}}, \mathbf{\Gamma})$ are required. The practical relevance of EI can be seen in sensor networks where information is transmitted over multiple hops and intermediate nodes fuse the received information. In this case, several nodes may share the same information, which leads to the decompositions (1) and (2). Using EI for fusion then circumvents the need to bookkeep the already incorporated information. In the following sections, the EI fusion formulae are reviewed, analyzed, and reformulated.

III. REVIEW OF ELLIPSOIDAL INTERSECTION

The ellipsoidal intersection (EI) method has been derived in [21] as a conservative fusion rule for estimates that share unknown common information. For the purpose of computing a consistent estimate irrespective of the common information $(\hat{\underline{\gamma}}, \mathbf{\Gamma})$, an estimate $(\hat{\underline{\gamma}}_{\mathrm{EI}}, \mathbf{\Gamma}_{\mathrm{EI}})$ with maximum possible $\mathbf{\Gamma}_{\mathrm{EI}}^{-1}$ is determined that can be subtracted from the fusion result (9) in place of the actual but unknown common estimate.

From the decompositions (1b) and (2b) of the corresponding covariance matrices, it follows that the unknown common information $(\hat{\underline{\gamma}}, \mathbf{\Gamma})$ has to obey the inequalities

$$\mathbf{C}_\mathsf{A} \leq \mathbf{\Gamma} \quad \text{and} \quad \mathbf{C}_\mathsf{B} \leq \mathbf{\Gamma} \ . \tag{10}$$

Consequently, $\mathbf{\Gamma}_{\mathrm{EI}}$ has to satisfy (10) and to be as small as possible in order to maximize its inverse. For this reason, the matrix $\mathbf{\Gamma}_{\mathrm{EI}}$ is designed to be the shape matrix of the minimum-volume ellipsoid that encloses the ellipsoids related to $\mathbf{C}_\mathsf{A}$ and $\mathbf{C}_\mathsf{B}$, i.e., (10) is reformulated to $\mathcal{E}(\underline{0}, \mathbf{C}_\mathsf{A}) \cup \mathcal{E}(\underline{0}, \mathbf{C}_\mathsf{B}) \subseteq \mathcal{E}(\underline{0}, \mathbf{\Gamma}_{\mathrm{EI}})$. As explained in [21], the smallest covering ellipsoid can be computed by means of a joint diagonalization

$$\mathbf{D}_\mathsf{A} = \mathbf{T}\mathbf{C}_\mathsf{A}\mathbf{T}^\mathrm{T} \quad \text{and} \quad \mathbf{D}_\mathsf{B} = \mathbf{T}\mathbf{C}_\mathsf{B}\mathbf{T}^\mathrm{T} \tag{11}$$

of $\mathbf{C}_\mathsf{A}$ and $\mathbf{C}_\mathsf{B}$ with an appropriate transformation matrix $\mathbf{T}$. By determining the component-wise maximum

$$(\bar{\mathbf{D}})_{ii} = \max\{(\mathbf{D}_\mathsf{A})_{ii}, (\mathbf{D}_\mathsf{B})_{ii}\} \ , \tag{12}$$

we obtain the desired common estimate matrix

$$\mathbf{\Gamma}_{\mathrm{EI}} = \mathbf{T}^{-1}\bar{\mathbf{D}}\mathbf{T}^{-\mathrm{T}} \ . \tag{13}$$

In order to compute $\hat{\underline{\gamma}}_{\mathrm{EI}}$, the explicit formula

$$\begin{aligned}\hat{\underline{\gamma}}_{\mathrm{EI}} = &\left(\mathbf{C}_\mathsf{A}^{-1} + \mathbf{C}_\mathsf{B}^{-1} - 2\mathbf{\Gamma}_{\mathrm{EI}}^{-1} + 2\eta\mathbf{I}\right)^{-1} \cdot \\ &\left(\left(\mathbf{C}_\mathsf{A}^{-1} - \mathbf{\Gamma}_{\mathrm{EI}}^{-1} + \eta\mathbf{I}\right)\hat{\underline{\mathbf{x}}}_\mathsf{A} + \left(\mathbf{C}_\mathsf{B}^{-1} - \mathbf{\Gamma}_{\mathrm{EI}}^{-1} + \eta\mathbf{I}\right)\hat{\underline{\mathbf{x}}}_\mathsf{B}\right)\end{aligned} \tag{14}$$

is employed in [21], where $\eta > 0$ is a regularization factor to avoid numerical instabilities. The parameters (13) and (14) enter (9) so as to obtain the fused mean $\hat{\underline{\mathbf{x}}}_{\mathrm{EI}}$ according to

$$\hat{\underline{\mathbf{x}}}_{\mathrm{EI}} = \mathbf{C}_{\mathrm{EI}}\left(\mathbf{C}_\mathsf{A}^{-1}\hat{\underline{\mathbf{x}}}_\mathsf{A} + \mathbf{C}_\mathsf{B}^{-1}\hat{\underline{\mathbf{x}}}_\mathsf{B} - \mathbf{\Gamma}_{\mathrm{EI}}^{-1}\hat{\underline{\gamma}}_{\mathrm{EI}}\right) \ . \tag{15a}$$

and the fused covariance matrix $\mathbf{C}_{\mathrm{EI}}$ according to

$$\mathbf{C}_{\mathrm{EI}}^{-1} = \mathbf{C}_\mathsf{A}^{-1} + \mathbf{C}_\mathsf{B}^{-1} - \mathbf{\Gamma}_{\mathrm{EI}}^{-1} \ . \tag{15b}$$

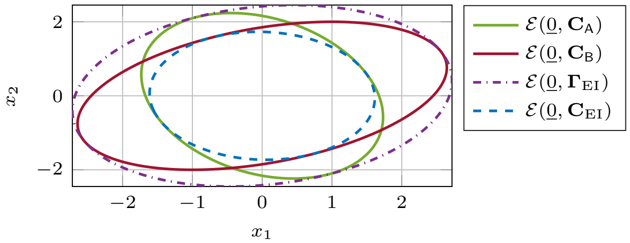


Fig. 2. Internal approximation of the intersection and external approximation of the union.

In the following section, an algebraic reformulation of the EI fusion formulae is proposed, which also reveals an important relationship to other fusion rules.

IV. Reformulations and Simplifications of Ellipsoidal Intersection

In this subsection, a reformulation of EI is discussed that avoids an explicit computation of the parameters $\hat{\underline{\gamma}}_\mathrm{EI}$ and $\mathbf{\Gamma}_\mathrm{EI}$. It turns out that the B/C fusion formulae can be employed to compute $(\hat{\underline{\mathbf{x}}}_\mathrm{EI}, \mathbf{C}_\mathrm{EI})$. In doing so, a specific correlation structure between the estimates to be fused is assumed.

A. Internal Ellipsoidal Approximation

In a first step, it is shown that $\mathbf{C}_\mathrm{EI}$ can directly be computed. The common matrix $\mathbf{\Gamma}_\mathrm{EI}$ in (13) has been designed as a minimum upper bound (12) on the covariance matrices. The joint diagonalization (11) can directly be applied to the fused covariance matrix (15b), which yields

$$\mathbf{D}_\mathrm{EI} = \mathbf{T}\mathbf{C}_\mathrm{EI}\mathbf{T}^\mathrm{T} = \left(\mathbf{D}_\mathrm{A}^{-1} + \mathbf{D}_\mathrm{B}^{-1} - \bar{\mathbf{D}}^{-1}\right)^{-1} \ .$$

This matrix has the diagonal components

$$\begin{aligned}(\mathbf{D}_\mathrm{EI})_{ii} &= \left(\tfrac{1}{(\mathbf{D}_\mathrm{A})_{ii}} + \tfrac{1}{(\mathbf{D}_\mathrm{B})_{ii}} - \tfrac{1}{\max\{(\mathbf{D}_\mathrm{A})_{ii}, (\mathbf{D}_\mathrm{B})_{ii}\}}\right)^{-1} \\ &= \min\left\{(\mathbf{D}_\mathrm{A})_{ii}, (\mathbf{D}_\mathrm{B})_{ii}\right\} \ ,\end{aligned} \tag{16}$$

Hence, $\mathbf{C}_\mathrm{EI}$ is computed by taking the minimum diagonal components instead of the maximum components as in (12), and $\mathbf{C}_\mathrm{EI}$ can be obtained without explicitly computing $\mathbf{\Gamma}_\mathrm{EI}$

Corollary 3: *The matrix $\mathbf{C}_\mathrm{EI}$ is the shape matrix of the Löwner-John ellipsoid $\mathcal{E}(\underline{0}, \mathbf{C}_\mathrm{EI})$, i.e., maximum-volume ellipsoid, contained in the intersection $\mathcal{E}(\underline{0}, \mathbf{C}_\mathrm{A}) \cap \mathcal{E}(\underline{0}, \mathbf{C}_\mathrm{B})$.*

Fig. 2 illustrates the relationship pointed out in the corollary. Such a construction of an internal ellipsoidal approximation has also been suggested by [19]. There, the computation of the fused mean has not been adapted, which is an important difference to EI. Also, the refinement of the internal approximation in [20] yields a mean different to EI. In the next step, it is shown that EI computes a mean that is consistent with fused covariance matrix $\mathbf{C}_\mathrm{EI}$.

B. Direct Computation of the Fusion Result

For the fused covariance matrix (15b), a direct computation, which avoids the calculation of (13), is already given by (16). The common mean (14) has been derived in [21] by considering the transformed vectors $\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{A}$ and $\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{B}$. In this transformed state space, the common mean (14) is equivalent to

$$\left(\mathbf{T}\hat{\underline{\gamma}}_\mathrm{EI}\right)_i = \begin{cases} (\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{A})_i & \text{if } (\mathbf{D}_\mathrm{A})_{ii} > (\mathbf{D}_\mathrm{B})_{ii} \ , \\ (\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{B})_i & \text{if } (\mathbf{D}_\mathrm{A})_{ii} < (\mathbf{D}_\mathrm{B})_{ii} \ , \\ \frac{1}{2}\left((\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{A})_i + (\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{B})_i\right) & \text{if } (\mathbf{D}_\mathrm{A})_{ii} = (\mathbf{D}_\mathrm{B})_{ii} \ . \end{cases} \tag{17}$$

The latter case also reveals the reason why the regularization in (14) is needed: Applying the joint diagonalization to the matrices in (14) it can be seen that the matrices become singular if $(\mathbf{D}_\mathrm{A})_{ii} = (\mathbf{D}_\mathrm{B})_{ii}$ holds for some diagonal entries.

Analogously, the fused mean (15a) can also be expressed in the transformed state space, which yields

$$\begin{aligned}\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{EI} &= \mathbf{T}\mathbf{C}_\mathrm{EI}\left(\mathbf{C}_\mathrm{A}^{-1}\hat{\underline{\mathbf{x}}}_\mathrm{A} + \mathbf{C}_\mathrm{B}^{-1}\hat{\underline{\mathbf{x}}}_\mathrm{B} - \mathbf{\Gamma}_\mathrm{EI}^{-1}\hat{\underline{\gamma}}_\mathrm{EI}\right) \\ &= \mathbf{T}\mathbf{C}_\mathrm{EI}\mathbf{T}^\mathrm{T}\mathbf{T}^{-\mathrm{T}}\left(\mathbf{C}_\mathrm{A}^{-1}\hat{\underline{\mathbf{x}}}_\mathrm{A} + \mathbf{C}_\mathrm{B}^{-1}\hat{\underline{\mathbf{x}}}_\mathrm{B} - \mathbf{\Gamma}_\mathrm{EI}^{-1}\hat{\underline{\gamma}}_\mathrm{EI}\right) \\ &= \mathbf{D}_\mathrm{EI}\left(\mathbf{D}_\mathrm{A}^{-1}\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{A} + \mathbf{D}_\mathrm{B}^{-1}\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{B} - \bar{\mathbf{D}}^{-1}\mathbf{T}\hat{\underline{\gamma}}_\mathrm{EI}\right) \ .\end{aligned}$$

By inserting (12), (16), and (17), we arrive at

$$\left(\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{EI}\right)_i = \begin{cases} (\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{B})_i & \text{if } (\mathbf{D}_\mathrm{A})_{ii} > (\mathbf{D}_\mathrm{B})_{ii} \ , \\ (\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{A})_i & \text{if } (\mathbf{D}_\mathrm{A})_{ii} < (\mathbf{D}_\mathrm{B})_{ii} \ , \\ \frac{1}{2}\left((\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{A})_i + (\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{B})_i\right) & \text{if } (\mathbf{D}_\mathrm{A})_{ii} = (\mathbf{D}_\mathrm{B})_{ii} \ . \end{cases} \tag{18}$$

Note the difference in the first two cases. This means that $\left(\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{EI}\right)_i$ consists of the element $(\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{A})_i$ or $(\mathbf{T}\hat{\underline{\mathbf{x}}}_\mathrm{B})_i$ depending on which one is attributed to the smaller diagonal element of $\mathbf{D}_\mathrm{A}$ and $\mathbf{D}_\mathrm{B}$. Consequently, the fused mean can be computed in the transformed state space without the need for explicitly calculating $\hat{\underline{\gamma}}_\mathrm{EI}$. This reformulation offers the advantage that no regularization factor has to be introduced.

Another possibility to compute the EI fusion result is related to the B/C formulae, as pointed out in the following considerations. For this purpose, we assume that the cross-covariance matrix (4) of the estimates to be fused is given by $\mathbf{C}_\mathrm{AB} = \mathbf{C}_\mathrm{A}\mathbf{\Gamma}_\mathrm{EI}^{-1}\mathbf{C}_\mathrm{B}$, which implies that $\mathbf{\Gamma}_\mathrm{EI}$ is assumed be the actual covariance matrix of the (unknown) common estimate in (1) and (2). It is an interesting observation that $\mathbf{C}_\mathrm{AB}$ is then symmetric and is given by

$$\mathbf{C}_\mathrm{AB} \overset{(4)}{=} \mathbf{C}_\mathrm{A}\mathbf{\Gamma}_\mathrm{EI}^{-1}\mathbf{C}_\mathrm{B} \overset{(13)}{=} \mathbf{T}^{-1}\mathbf{D}_\mathrm{A}\,\bar{\mathbf{D}}^{-1}\,\mathbf{D}_\mathrm{B}\mathbf{T}^{-\mathrm{T}} = \mathbf{C}_\mathrm{EI} \ , \tag{19}$$

where the latter equation follows from

$$\begin{aligned}(\mathbf{D}_\mathrm{A})_{ii}\,(\bar{\mathbf{D}})_{ii}^{-1}\,(\mathbf{D}_\mathrm{B})_{ii} &= \tfrac{(\mathbf{D}_\mathrm{A})_{ii}(\mathbf{D}_\mathrm{B})_{ii}}{\max\{(\mathbf{D}_\mathrm{A})_{ii}, (\mathbf{D}_\mathrm{B})_{ii}\}} \\ &= \min\left\{(\mathbf{D}_\mathrm{A})_{ii}, (\mathbf{D}_\mathrm{B})_{ii}\right\} \ .\end{aligned}$$

Instead of combining the estimates with the EI fusion rule (15), we apply the B/C fusion rule (5) by employing the assumed correlation structure (19). The fusion result (5) then becomes

$$\hat{\underline{\mathbf{x}}}_\mathrm{B/C} \overset{(5a)}{=} \mathbf{K}_\mathrm{B/C}\,\hat{\underline{\mathbf{x}}}_\mathrm{A} + \mathbf{L}_\mathrm{B/C}\,\hat{\underline{\mathbf{x}}}_\mathrm{B} \tag{20a}$$

and

$$\begin{aligned}\mathbf{C}_\mathrm{B/C} \overset{(5b)}{=} &\ \mathbf{K}_\mathrm{B/C}\mathbf{C}_\mathrm{A}\mathbf{K}_\mathrm{B/C}^\mathrm{T} + \mathbf{L}_\mathrm{B/C}\mathbf{C}_\mathrm{B}\mathbf{L}_\mathrm{B/C}^\mathrm{T} + \\ &\ \mathbf{K}_\mathrm{B/C}\mathbf{C}_\mathrm{EI}\mathbf{L}_\mathrm{B/C}^\mathrm{T} + \mathbf{L}_\mathrm{B/C}\mathbf{C}_\mathrm{EI}\mathbf{K}_\mathrm{B/C}^\mathrm{T} \ ,\end{aligned} \tag{20b}$$

where the gains $\mathbf{K}_{\mathrm{B/C}}$ and $\mathbf{L}_{\mathrm{B/C}} = \mathbf{I} - \mathbf{K}_{\mathrm{B/C}}$ are given by

$$\mathbf{K}_{\mathrm{B/C}} \overset{(6a)}{=} \left(\mathbf{C}_{\mathsf{B}} - \mathbf{C}_{\mathrm{EI}}\right) \cdot \left(\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}} - 2\mathbf{C}_{\mathrm{EI}}\right)^{-1} \quad (21a)$$

$$\mathbf{L}_{\mathrm{B/C}} \overset{(6b)}{=} \left(\mathbf{C}_{\mathsf{A}} - \mathbf{C}_{\mathrm{EI}}\right) \cdot \left(\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}} - 2\mathbf{C}_{\mathrm{EI}}\right)^{-1} \quad (21b)$$

Although Theorem 1 leads us to expect that (20) should be different from (15), the following theorem proves us wrong.

Theorem 4: *The combination* (20) *yields the same result as the* EI *fusion rule* (15).

Proof: The proof of this theorem can be found in Appendix B. □

Remark 5: *In order to circumvent numerical instabilities, the gains* (21) *can be modified according to*

$$\mathbf{K}_{\mathrm{B/C}} = \left(\mathbf{C}_{\mathsf{B}} - \widetilde{\mathbf{C}}_{\mathrm{EI}}\right) \cdot \left(\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}} - 2\widetilde{\mathbf{C}}_{\mathrm{EI}}\right)^{-1} \quad (22a)$$

$$\mathbf{L}_{\mathrm{B/C}} = \left(\mathbf{C}_{\mathsf{A}} - \widetilde{\mathbf{C}}_{\mathrm{EI}}\right) \cdot \left(\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}} - 2\widetilde{\mathbf{C}}_{\mathrm{EI}}\right)^{-1} \quad (22b)$$

with $\widetilde{\mathbf{C}}_{\mathrm{EI}} = \mathbf{C}_{\mathrm{EI}} + \eta \mathbf{I}\,,\ \eta > 0$ *as it has been done in* (14).

The fusion rule (20) offers the advantage that (13) and (14) do not have to be explicitly computed. The EI fusion rule is equivalent to the B/C fusion rule when it is assumed that the estimates have the cross-covariance matrix (19). It is an interesting observation that that the fused covariance matrix (20b) is equal to the assumed cross-covariance matrix (19). Both the original formulation of EI and its reformulation in terms of the B/C formulae are prone to numerical instabilities. An alternative is the direct computation of the fused mean and covariance matrix by (18) and (16) in the transformed state space. Matlab functions of the presented reformulations can be downloaded from `www.bennoack.net/EI`.

The presented reformulations of EI prove to be useful for efficient implementations of the algorithm. However, it remains an open question under which conditions EI can be considered to constitute a consistent fusion rule. This question is addressed in [27] where also a different parameterization of the common estimate with improved consistency is proposed.

V. CONCLUSIONS

The EI fusion rule has been proposed as an alternative to CI and yields less conservative results. EI is tailored to a specific correlation structure that originates from an unknown common estimate shared by the estimates to be fused. With the aim of conservativeness, EI computes the common estimate with the smallest possible covariance matrix, i.e. maximum inverse covariance matrix, that can be subtracted from the fusion result.

The estimate provided by the EI fusion rule is related to a maximum internal ellipsoid of the intersection of the covariance ellipsoids that correspond to the estimates to be fused. The considerations in this paper revealed that EI can be computed with the aid of the B/C fusion rule. In doing so, it is implicitly assumed that the considered estimates share a common estimate with covariance matrix $\mathbf{\Gamma}_{\mathrm{EI}}$. This parameterization constitutes an important special case of Theorem 1 for which equality holds. Employing the B/C rule offers the advantage that the parameters of the common estimate do not need to be computed explicitly.

APPENDIX

A Proof of Theorem 1

The inequality (8) to be considered is equivalent to

$$\mathbf{C}_{\mathrm{opt}}^{-1} \geq \mathbf{C}_{\mathrm{B/C}}^{-1} \quad (A.1)$$

In order to prove this inequality, it must be shown that the difference $\mathbf{C}_{\mathrm{opt}}^{-1} - \mathbf{C}_{\mathrm{B/C}}^{-1}$ is positive (semi-)definite. As for instance stated in [28], the B/C fusion rule can be rewritten into

$$\begin{aligned}\mathbf{C}_{\mathrm{B/C}}^{-1} &= \begin{bmatrix}\mathbf{I} & \mathbf{I}\end{bmatrix} \begin{bmatrix}\mathbf{C}_{\mathsf{A}} & \mathbf{C}_{\mathsf{AB}} \\ \mathbf{C}_{\mathsf{BA}} & \mathbf{C}_{\mathsf{B}}\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{I} \\ \mathbf{I}\end{bmatrix} \\ &= \mathbf{C}_{\mathsf{A}}^{-1} + \left(\mathbf{I} - \mathbf{C}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{AB}}\right)\left(\mathbf{C}_{\mathsf{B}} - \mathbf{C}_{\mathsf{BA}}\mathbf{C}_{\mathsf{A}}^{-1}\mathbf{C}_{\mathsf{AB}}\right)^{-1} \\ &\qquad \left(\mathbf{I} - \mathbf{C}_{\mathsf{BA}}\mathbf{C}_{\mathsf{A}}^{-1}\right)\,.\end{aligned}$$

By exploiting the specific correlation structure (4), this matrix can be further rewritten to

$$\begin{aligned}\mathbf{C}_{\mathrm{B/C}}^{-1} &= \mathbf{C}_{\mathsf{A}}^{-1} + \left(\mathbf{I} - \mathbf{\Gamma}^{-1}\mathbf{C}_{\mathsf{B}}\right)\left(\mathbf{C}_{\mathsf{B}} - \mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}^{-1}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}^{-1}\mathbf{C}_{\mathsf{B}}\right)^{-1} \\ &\qquad \left(\mathbf{I} - \mathbf{C}_{\mathsf{B}}\mathbf{\Gamma}^{-1}\right)\,. \\ &= \mathbf{C}_{\mathsf{A}}^{-1} + \left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\right)\left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}^{-1}\right)^{-1} \\ &\qquad \left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\right)\,.\end{aligned}$$

With (9b), the inequality (A.1) becomes

$$\begin{aligned}\left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\right) &\geq \left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\right) \cdot \\ &\left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}^{-1}\right)^{-1}\left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\right)\,.\end{aligned}$$

By pre- und post-multiplying with $(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1})^{-1}$, it remains to be shown that

$$\left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\right)^{-1} \geq \left(\mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}^{-1}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}^{-1}\right)^{-1}$$

holds. This follows from $\mathbf{\Gamma}^{-1}\mathbf{C}_{\mathsf{A}}\mathbf{\Gamma}^{-1} \leq \mathbf{\Gamma}^{-1}$, which is a consequence of $\mathbf{C}_{\mathsf{A}} \leq \mathbf{\Gamma}$ in (10).

B Proof of Theorem 4

In order to show the equivalence of the formulae (15) and (20), we first analyze (20a) by using the transformation

$$\begin{aligned}\mathbf{T}\underline{\hat{\mathbf{x}}}_{\mathrm{B/C}} &= \mathbf{T}\mathbf{K}_{\mathrm{B/C}}\underline{\hat{\mathbf{x}}}_{\mathsf{A}} + \mathbf{T}\mathbf{K}_{\mathrm{B/C}}\underline{\hat{\mathbf{x}}}_{\mathsf{B}} \\ &= \bar{\mathbf{K}}_{\mathrm{B/C}}\mathbf{T}\underline{\hat{\mathbf{x}}}_{\mathsf{A}} + \bar{\mathbf{L}}_{\mathrm{B/C}}\mathbf{T}\underline{\hat{\mathbf{x}}}_{\mathsf{B}}\end{aligned} \quad (B.1)$$

with

$$\begin{aligned}\bar{\mathbf{K}}_{\mathrm{B/C}} &= \mathbf{T}\mathbf{K}_{\mathrm{B/C}}\mathbf{T}^{-1} \\ &= \mathbf{T}\left(\mathbf{C}_{\mathsf{B}} - \mathbf{C}_{\mathrm{EI}}\right) \cdot \left(\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}} - 2\mathbf{C}_{\mathrm{EI}}\right)^{-1}\mathbf{T}^{-1} \\ &= \mathbf{T}\left(\mathbf{C}_{\mathsf{B}} - \mathbf{C}_{\mathrm{EI}}\right)\mathbf{T}^{\mathrm{T}}\mathbf{T}^{-\mathrm{T}}\left(\mathbf{C}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}} - 2\mathbf{C}_{\mathrm{EI}}\right)^{-1}\mathbf{T}^{-1} \\ &= \left(\mathbf{D}_{\mathsf{B}} - \mathbf{D}_{\mathrm{EI}}\right) \cdot \left(\mathbf{D}_{\mathsf{A}} + \mathbf{D}_{\mathsf{B}} - 2\mathbf{D}_{\mathrm{EI}}\right)^{-1}\end{aligned}$$

and

$$\bar{\mathbf{L}}_{\mathrm{B/C}} = \left(\mathbf{D}_{\mathsf{A}} - \mathbf{D}_{\mathrm{EI}}\right) \cdot \left(\mathbf{D}_{\mathsf{A}} + \mathbf{D}_{\mathsf{B}} - 2\mathbf{D}_{\mathrm{EI}}\right)^{-1}\,.$$

With the definition (16) of $\mathbf{D}_{\mathrm{EI}}$, the components of these diagonal gains are

$$\left(\bar{\mathbf{K}}_{\mathrm{B/C}}\right)_{ii} = \begin{cases} 0 & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} > (\mathbf{D}_{\mathsf{B}})_{ii}\,, \\ 1 & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} < (\mathbf{D}_{\mathsf{B}})_{ii}\,, \\ * & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} = (\mathbf{D}_{\mathsf{B}})_{ii}\,. \end{cases}$$

and

$$\left(\bar{\mathbf{L}}_{\mathrm{B/C}}\right)_{ii} = \begin{cases} 1 & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} > (\mathbf{D}_{\mathsf{B}})_{ii} \ , \\ 0 & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} < (\mathbf{D}_{\mathsf{B}})_{ii} \ , \\ * & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} = (\mathbf{D}_{\mathsf{B}})_{ii} \ . \end{cases} \tag{B.2}$$

For the first two cases, it becomes apparent that (B.1) is equal to (18). The third case $(\mathbf{D}_{\mathsf{A}})_{ii} = (\mathbf{D}_{\mathsf{B}})_{ii}$ deserves special attention because of a division by zero. As it has been done in (14), a regularization matrix $\eta\mathbf{I}$ with small $\eta > 0$ can be added to $\mathbf{D}_{\mathrm{EI}}$. In this case, the entry $*$ becomes $\frac{1}{2}$, and the equality can also be seen for the third case. The regularized gains (22) are hence used to treat numerical instabilities.

To show equality for the fused covariance matrices (15b) and (20b), the latter matrix is written as

$$\mathbf{C}_{\mathrm{B/C}} \overset{(5b)}{=} \mathbf{C}_{\mathsf{A}} - \mathbf{L}_{\mathrm{B/C}}\big(\mathbf{C}_{\mathsf{A}} - \mathbf{C}_{\mathrm{EI}}\big) \ .$$

We apply the transformation matrix $\mathbf{T}$ from (11) and obtain

$$\begin{aligned} \mathbf{D}_{\mathrm{B/C}} &= \mathbf{T}\mathbf{C}_{\mathrm{B/C}}\mathbf{T}^{\mathrm{T}} \\ &= \mathbf{T}\mathbf{C}_{\mathsf{A}}\mathbf{T}^{\mathrm{T}} - \mathbf{T}\mathbf{L}_{\mathrm{B/C}}\big(\mathbf{C}_{\mathsf{A}} - \mathbf{C}_{\mathrm{EI}}\big)\mathbf{T}^{\mathrm{T}} \\ &= \mathbf{D}_{\mathsf{A}} - \bar{\mathbf{L}}_{\mathrm{B/C}}\big(\mathbf{D}_{\mathsf{A}} - \mathbf{D}_{\mathrm{EI}}\big) \ . \end{aligned}$$

By taking into account the result (16) and using (B.2), the diagonal components yield

$$\left(\mathbf{D}_{\mathrm{B/C}}\right)_{ii} = \begin{cases} (\mathbf{D}_{\mathsf{B}})_{ii} & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} > (\mathbf{D}_{\mathsf{B}})_{ii} \ , \\ (\mathbf{D}_{\mathsf{A}})_{ii} & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} < (\mathbf{D}_{\mathsf{B}})_{ii} \ , \\ (\mathbf{D}_{\mathsf{A}})_{ii} & \text{if } (\mathbf{D}_{\mathsf{A}})_{ii} = (\mathbf{D}_{\mathsf{B}})_{ii} \ , \end{cases}$$

which is the minimum (16) of the diagonal components of $\mathbf{D}_{\mathsf{A}}$ and $\mathbf{D}_{\mathsf{B}}$. Thus, $\mathbf{C}_{\mathrm{B/C}} = \mathbf{C}_{\mathrm{EI}}$ holds, and the combination (20) is equal to the EI fusion rule (15).

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