Progressive Closed-Loop Chance-Constrained Control

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***Abstract*—Chance-constrained control is a difficult problem even if the considered system dynamics are linear. The difficulty stems from the facts that the chance constraints are difficult to evaluate and that the control law is nonlinear due to the constraints. In this paper, we present a novel approach to chance-constrained control, where we solve the unconstrained control problem first and then use a progressive method to gradually introduce the chance constraints. This has significant advantages compared to traditional methods, because we do not require a feasible initial solution for the numerical optimization algorithm. Finally, we evaluate the novel method and compare it to an approach from literature.**

***Keywords*—optimization, affine controller, barrier methods, scenario approximation**

I. Introduction

In many control applications, it is necessary to enforce constraints in order to guarantee a safe system operation [1]. On the one hand, input constraints restrict the possible values of the control input, e.g., the maximum steering angle of a vehicle or the maximum thrust of a plane. These types of constraints have to be considered when computing the optimal control input, but they can always be guaranteed because the control input is deterministically chosen by the controller. On the other hand, state constraints affect the state vector or its trajectory, e.g., the location of a vehicle or the amount of fuel used. In stochastic control, state constraints are typically harder to enforce because the state vector is affected by nondeterministic disturbances.

In the presence of unbounded stochastic uncertainties, it is, in general, impossible to guarantee that state constraints are never violated. A large disturbance, however unlikely, can never be completely ruled out, and it is thus not possible to give any hard bounds on the values of future state vectors. As a result, the field of *chance-constrained control* has emerged, where constraints only need to be satisfied with a certain probability (see Fig. 1).

However, chance-constrained control is not a trivial problem even in fairly simple scenarios, i.e., a linear system with state feedback where the state vector is directly accessible, convex polytopic constraints, and independent and identically distributed zero-mean white Gaussian noise. The unconstrained problem is easy to solve because the controller is linear with respect to the state, but the chance-constrained control problem exhibits additional difficulties. The main issues are that (i) the evaluation of the expected constraint violation is not possible in closed form because it requires computation of a multivariate integral, and that (ii) the resulting controller is nonlinear with respect to the state, and thus, difficult to compute.

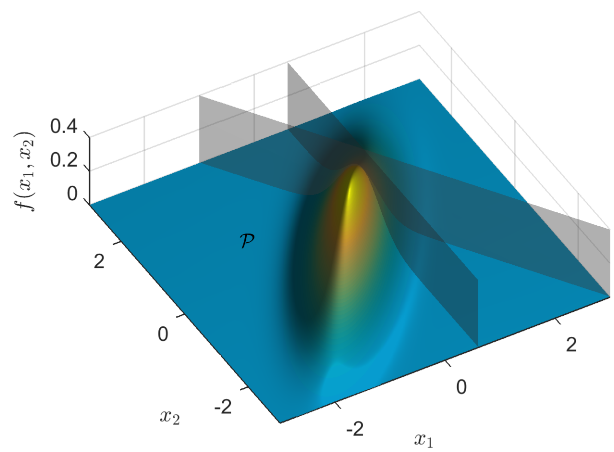

Fig. 1: Visualization of the concept of chance constraints. The state $[x_1, x_2]^T$ is distributed according to a Gaussian density. It has to satisfy two linear constraints (visualized as semitransparent walls) with a predefined probability $1-\alpha$. That means that the probability mass contained in the admissible region $\mathcal{P}$ has to exceed the threshold $1-\alpha$.

For the case where the system state is Gaussian and the admissible region is convex, problem (i) has been addressed in literature by a number of authors [2]. For example, van Hessem et al. [3], [4], [5] have proposed a method where the probability mass of the state that has to satisfy the constraints is approximated by an ellipsoid. The controller is then chosen such that this ellipsoid is entirely contained within the constraints. Other approaches compute the constraint violation probability by approximating the admissible region, which is a polytope in these works, using axis-aligned boxes [6] or spherical sectors [7]. In both cases, the probability mass inside such a box or a sector is given in an analytical form. Other approaches include risk allocation [8], [9], [10] and Monte-Carlo methods that rely on stochastic sampling [11], [12], [13], [14], [15]. The latter are suitable for non-Gaussian states and non-convex admissible regions [2]. However, they ae typically computationally very intensive because a very high number of samples may be necessary.

In this paper, we address problem (ii). The contribution can be summarized as follows. Instead of an arbitrary nonlinear controller, we assume an affine controller, i.e., a linear controller with an additional offset parameter (similar to [16]). First, we solve the unconstrained control problem without the offset parameter using Linear Quadratic Gaussian control (LQG) [17]. Then, we introduce the constraints gradually into the corresponding optimization problem using progressive methods. These types of methods have previously been used in a number of applications in filtering [18], [19], [20], [21]. The key advantage of progressive methods is that it is not necessary to have a feasible initial guess for the offset parameter, because the solution to the first problem in the progression is either known or very easy to obtain. Furthermore, we are able to consider the closed-loop control problem in the proposed approach, which is an advantage compared to open-loop MPC-based solutions such as [22]. Also, the proposed method does not require feedback of the empirical constraint violation probability (unlike e.g., [23]), which is advantageous in some situations, e.g., if the constraint violation probability is very small.

This paper is structured as follows. In Sec. II, we give a rigorous statement of the problem under consideration. Then, we present the proposed approach in Sec. III and derive all required equations. Subsequently, we evaluate the novel method using simulations in Sec. IV, where we illustrate the advantages of the proposed approach by comparing it to a state-of-the-art method. Finally, we form a conclusion in Sec. V.

We use the following notation. Matrices, e.g., $\mathbf{A}, \mathbf{B}$ are written in bold capital letters, and vectors, e.g., $\underline{x}, \underline{y}$ are underlined. Sequences $a_1, \ldots, a_n$ are abbreviated as $a_{1:n}$. The expected value of a random variable x is written as $\mathbb{E}\left(x\right)$ and the probability of an event ω is written as $\mathbb{P}(\omega)$.

II. Problem Formulation

In this paper, we consider state-feedback control of a linear stochastic system with dynamics

$$\underline{x}_{k+1} = \mathbf{A}_k \underline{x}_k + \mathbf{B}_k \underline{u}_k + \underline{w}_k \ , \tag{1}$$

where $\underline{x}_k \in \mathbb{R}^n$ is the state at time step k, $\underline{u}_k \in \mathbb{R}^m$ is the control input, and $w_k \in \mathbb{R}^n$ is zero-mean independent and identically distributed Gaussian noise with covariance matrix $\mathbf{\Sigma}_k^w = \mathbb{E}\left(\underline{w}_k \underline{w}_k^T\right) \in \mathbb{R}^{n\times n}$. The initial state is given by $\underline{x}_0$.

Furthermore, we consider the quadratic finite-horizon cost function

$$J = \mathbb{E}\left(\underline{x}_K^T \mathbf{Q}_K \underline{x}_K + \sum_{k=0}^{K-1} (\underline{x}_k^T \mathbf{Q}_k \underline{x}_k + \underline{u}_k^T \mathbf{R}_k \underline{u}_k)\right) \ ,$$

where $K \in \mathbb{N}$ is the planning horizon, $\mathbf{Q}_k \in \mathbb{R}^{n\times n}$ is positive semidefinite, and $\mathbf{R}_k \in \mathbb{R}^{m\times m}$ is positive definite.

The chance constraints are given by the inequality

$$\mathbb{P}(\underline{x}_{0:K} \in \mathcal{P}) \geq 1 - \alpha \ , \tag{2}$$

where $\mathcal{P} \subset \mathbb{R}^{n\cdot(K+1)}$ is a convex polytope determined by an intersection of a finite number g of halfspaces in $\mathbb{R}^{n\cdot(K+1)}$ and $\alpha \in (0, 0.5)$ is the maximum constraint violation probability. We parameterize the admissible set as

$$\mathcal{P} = \{\underline{x} \in \mathbb{R}^{n\cdot(K+1)} : \mathbf{F}\cdot[\underline{x}_0^T, \ldots, \underline{x}_K^T]^T \leq \underline{b}\} \ .$$

Each row of $\mathbf{F} \in \mathbb{R}^{g\times(n\cdot(K+1))}$ together with the corresponding entry of $\underline{b} \in \mathbb{R}^g$ defines a hyperplane in Hessian normal form.

The described problem can be summarized as the optimization problem

$$\begin{aligned} \min_{\underline{u}_{0:K-1}} \ & J \\ \text{s.t.} \ & (1), (2), \\ & \widehat{\underline{x}}_0 = \underline{x}_0, \ \underline{w}_k \sim \mathcal{N}(\underline{0}, \mathbf{\Sigma}_k^w) \ , \end{aligned} \tag{3}$$

where we seek to obtain a sequence of control inputs that minimizes the expected cost over the planning horizon.

In the following, we assume an affine controller, i.e., the control input is given by

$$\underline{u}_k = \mathbf{L}_k \underline{x}_k + \underline{r}_k \ , \tag{4}$$

where $\mathbf{L}_k$ is the regulator gain and $\underline{r}_k$ is an offset representing the feed-forward part. The regulator gain $\mathbf{L}_k$ attenuates disturbances $\underline{w}_k$ and stabilizes the system (1), while the offset $\underline{r}_k$ drives the state such that it satisfies the chance constraints.

Using (4), the system equation (1) can be rewritten to obtain the closed-loop system dynamics according to

$$\underline{x}_{k+1} = (\underbrace{\mathbf{A}_k + \mathbf{B}_k \mathbf{L}_k}_{=:\widehat{\mathbf{A}}_k})\underline{x}_k + \mathbf{B}_k \underline{r}_k + \underline{w}_k \ . \tag{5}$$

Thus, the mean of $\underline{x}_{k+1}$ can be propagated through the closed-loop system using the equation

$$\widehat{\underline{x}}_{k+1} = \widehat{\mathbf{A}}_k \widehat{\underline{x}}_k + \mathbf{B}_k \underline{r}_k \ ,$$

and the second moment $\mathbf{X}_k = \mathbb{E}\left(\underline{x}_k \underline{x}_k^T\right)$ can be obtained as

$$\begin{aligned} \mathbf{X}_{k+1} = \ & \widehat{\mathbf{A}}_k \mathbf{X}_k \widehat{\mathbf{A}}_k^T + \widehat{\mathbf{A}}_k \widehat{\underline{x}}_k \underline{r}_k \mathbf{B}_k^T \\ & + \mathbf{B}_k \underline{r}_k \widehat{\underline{x}}_k^T \widehat{\mathbf{A}}_k^T + \mathbf{B}_k \underline{r}_k \underline{r}_k^T \mathbf{B}_k^T + \mathbf{\Sigma}_k^w \ . \end{aligned}$$

The dynamics of the second central moment, i.e., the covariance matrix $\bar{\mathbf{X}}_k = \mathbb{E}\left((\underline{x}_k - \widehat{\underline{x}}_k)(\underline{x}_k - \widehat{\underline{x}}_k)^T\right)$, are then given by

$$\begin{aligned} \bar{\mathbf{X}}_{k+1} &= \mathbf{X}_{k+1} - \widehat{\underline{x}}_{k+1} \widehat{\underline{x}}_{k+1}^T \\ &= \widehat{\mathbf{A}}_k \bar{\mathbf{X}}_k \widehat{\mathbf{A}}_k^T + \mathbf{\Sigma}_k^w \ . \end{aligned}$$

Now, we can rewrite the cost function J using (4) to eliminate the dependence on $\underline{u}_0, \ldots, \underline{u}_{K-1}$, which results in

$$\begin{aligned} J = \ & \mathbb{E}\left(\underline{x}_K^T \mathbf{Q}_K \underline{x}_K\right) + \sum_{k=0}^{K-1} \Big(\mathbb{E}\left(\underline{x}_k^T \mathbf{Q}_k \underline{x}_k\right) \\ & + \mathbb{E}\left((\mathbf{L}_k \underline{x}_k + \underline{r}_k)^T \mathbf{R}_k (\mathbf{L}_k \underline{x}_k + \underline{r}_k)\right)\Big) \\ = \ & \mathbb{E}\left(\underline{x}_K^T \mathbf{Q}_K \underline{x}_K\right) + \sum_{k=0}^{K-1} \Big(\mathbb{E}\left(\underline{x}_k^T \mathbf{Q}_k \underline{x}_k\right) \\ & + \mathbb{E}\left(\underline{x}_k^T \mathbf{L}_k^T \mathbf{R}_k \mathbf{L}_k \underline{x}_k + 2\underline{x}_k^T \mathbf{L}_k^T \mathbf{R}_k + \underline{r}_k^T \mathbf{R}_k \underline{r}_k\right)\Big) \ . \end{aligned}$$

Then, we are able to reformulate the optimization problem (3) according to

$$\begin{aligned} \min_{\mathbf{L}_{0:k-1}, \underline{r}_{0:K-1}} \ & J \\ \text{s.t.} \ & (5), (2), \\ & \widehat{\underline{x}}_0 = \underline{x}_0, \ \underline{w}_k \sim \mathcal{N}(\underline{0}, \mathbf{\Sigma}_k^w) \ , \end{aligned}$$

i.e., we optimize with respect to $\mathbf{L}_{0:K-1}$ and $\underline{r}_{0:K-1}$ now.

III. Proposed Approach

The key idea of the proposed approach consists in solving the unconstrained control problem first and then progressively introducing the chance constraints in order to find a solution to the chance-constrained control problem. Optimizing with respect to $\mathbf{L}_{0:k-1}$ as well as $\underline{r}_{0:K-1}$ leads to a very high-dimensional optimization problem, which is possible but computationally expensive to solve. To avoid this problem, we use an approximation and obtain $\mathbf{L}_k$ as the LQG solution [17] of the unconstrained problem, i.e., we choose $\mathbf{L}_k$ such that it stabilizes the state using the LQG gain, and then optimize with respect to $\underline{r}_{0:K-1}$ only. Of course, $\mathbf{L}_k$ can be obtained by other means, e.g., such that the H_2 or the H_∞ norm of the state in the system without chance constraints is minimized.

A. State Augmentation

In order to rewrite the control problem, we consider the augmented state where the vectors and matrices encompass the entire planning horizon. For this purpose, we introduce the augmented vectors

$$\begin{aligned}\underline{\tilde{x}} &= [\underline{x}_0^T, \ldots, \underline{x}_K^T]^T\\ \underline{\tilde{r}} &= [\underline{r}_0^T, \ldots, \underline{r}_{K-1}^T]^T\\ \underline{\tilde{w}} &= [\underline{w}_0^T, \ldots, \underline{w}_{K-1}^T]^T \ ,\end{aligned}$$

and the augmented matrices

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{I} \\ \widehat{\mathbf{A}}_0 \\ \vdots \\ \widehat{\mathbf{A}}_{K-1} \cdot \ldots \cdot \widehat{\mathbf{A}}_0 \end{bmatrix} \ ,$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{B}_0 & \mathbf{0} & \ldots & \mathbf{0} \\ \widehat{\mathbf{A}}_1 \mathbf{B}_0 & \mathbf{B}_1 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\mathbf{A}}_{K-1} \cdot \ldots \cdot \widehat{\mathbf{A}}_1 \mathbf{B}_0 & \widehat{\mathbf{A}}_{K-1} \cdot \ldots \cdot \widehat{\mathbf{A}}_2 \mathbf{B}_1 & \ldots & \mathbf{B}_{K-1} \end{bmatrix} \ ,$$

$$\tilde{\mathbf{G}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} \\ \widehat{\mathbf{A}}_1 & \mathbf{I} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\mathbf{A}}_{K-1} \cdot \ldots \cdot \widehat{\mathbf{A}}_1 & \widehat{\mathbf{A}}_{K-1} \cdot \ldots \cdot \widehat{\mathbf{A}}_2 & \ldots & \mathbf{I} \end{bmatrix} \ ,$$

which allow us to rewrite the closed-loop system equation given in (5) according to

$$\underline{\tilde{x}} = \tilde{\mathbf{A}} \underline{x}_0 + \tilde{\mathbf{B}} \underline{\tilde{r}} + \tilde{\mathbf{G}} \underline{\tilde{w}} \ . \tag{6}$$

Furthermore, we introduce the augmented matrices

$$\begin{aligned}\tilde{\mathbf{R}} &= \operatorname{diag}(\mathbf{R}_0, \ldots, \mathbf{R}_{K-1}) \ ,\\ \tilde{\mathbf{Q}} &= \operatorname{diag}(\mathbf{Q}_0 + \mathbf{L}_0^T \mathbf{R}_0 \mathbf{L}_0, \ldots,\\ &\qquad \mathbf{Q}_{K-1} + \mathbf{L}_{K-1}^T \mathbf{R}_{K-1} \mathbf{L}_{K-1}, \mathbf{Q}_K) \ ,\\ \tilde{\mathbf{L}} &= \operatorname{diag}(\mathbf{L}_0, \ldots, \mathbf{L}_{K-1}) \ ,\end{aligned}$$

which we use to rewrite the cost function as

$$\begin{aligned}\tilde{J} &= \mathbb{E}\left(\underline{\tilde{x}}^T \tilde{\mathbf{Q}} \underline{\tilde{x}} + \underline{\tilde{r}}^T \tilde{\mathbf{R}} \underline{\tilde{r}} + 2 \underline{\tilde{x}} \tilde{\mathbf{L}} \tilde{\mathbf{R}} \underline{\tilde{r}}\right)\\ &= \mathbb{E}\left(\operatorname{tr}(\tilde{\mathbf{Q}} \underline{\tilde{x}} \underline{\tilde{x}}^T)\right) + \underline{\tilde{r}}^T \tilde{\mathbf{R}} \underline{\tilde{r}} + 2\mathbb{E}\left(\underline{\tilde{x}} \tilde{\mathbf{L}} \tilde{\mathbf{R}} \underline{\tilde{r}}\right)\\ &= \operatorname{tr}\left(\tilde{\mathbf{Q}} \tilde{\mathbf{A}} \mathbf{X}_0 \tilde{\mathbf{A}}^T + 2 \tilde{\mathbf{Q}} \tilde{\mathbf{A}} \underline{\hat{x}}_0 \underline{\tilde{r}}^T \tilde{\mathbf{B}}^T + \tilde{\mathbf{Q}} \tilde{\mathbf{B}} \underline{\tilde{r}} \underline{\tilde{r}}^T \tilde{\mathbf{B}}^T\right.\\ &\quad \left. + \tilde{\mathbf{Q}} \tilde{\mathbf{G}} \underline{\tilde{w}} \tilde{\mathbf{G}}^T\right) + \underline{\tilde{r}}^T \tilde{\mathbf{R}} \underline{\tilde{r}} + 2 \underline{\hat{x}}_0^T \tilde{\mathbf{A}} \tilde{\mathbf{L}}^T \tilde{\mathbf{R}} \underline{\tilde{r}} + 2 \underline{\tilde{r}}^T \tilde{\mathbf{B}}^T \tilde{\mathbf{L}}^T \tilde{\mathbf{R}} \underline{\tilde{r}} \ .\end{aligned}$$

Based on this result, we can compute the expected costs for a given $\underline{\tilde{r}}$ in closed form.

B. Scenario Approximation

Furthermore, we consider the chance constraint (2) for the joint probability density $f(\cdot)$ of $\underline{\tilde{x}}$

$$\mathbb{P}(\underline{x}_{0:K} \in \mathcal{P}) = \int_{\underline{\tilde{x}} \in \mathcal{P}} f(\underline{\tilde{x}}) d\underline{\tilde{x}} \geq 1 - \alpha \ ,$$

which contains an integral that is in general impossible to evaluate in closed form, but whose solution is required at each iteration step of the numerical optimization algorithm that minimizes $\tilde{J}$ with respect to $\underline{\tilde{r}}$. To address this problem, we approximate the probability density $f(\underline{\tilde{x}})$ with a predefined number of s samples, i.e.,

$$f(\underline{\tilde{x}}) \approx \frac{1}{s} \sum_{j=1}^{s} \delta(\underline{\tilde{x}} - \underline{d}_j) \ .$$

Note that the locations of the samples implicitly depend on $\underline{\tilde{r}}$. Using scenario approximation, the integral simplifies to a finite sum according to

$$\mathbb{P}(\underline{\tilde{x}}_{0:K} \in \mathcal{P}) = \int_{\underline{\tilde{x}} \in \mathcal{P}} f(\underline{\tilde{x}}) d\underline{\tilde{x}} \approx \frac{1}{s} \sum_{j=1}^{s} \mathbf{1}_{\mathcal{P}}(\underline{d}_j) \ ,$$

where $\mathbf{1}_{\mathcal{P}}(\cdot)$ is the indicator function of the set $\mathcal{P}$, which is defined as

$$\mathbf{1}_{\mathcal{P}}(\underline{\tilde{x}}) = \begin{cases} 1 \ , & \underline{\tilde{x}} \in \mathcal{P} \\ 0 \ , & \underline{\tilde{x}} \notin \mathcal{P} \end{cases} \ .$$

In this paper, we use stochastic sampling from a multivariate Gaussian density to obtain the sample positions $\underline{d}_1, \ldots, \underline{d}_s$ [24, Ch. XI]. This approach is similar to the Monte Carlo methods proposed in [13], [14], [11]. Note that other methods to approximate the chance constraints, e.g., box approximations from [6], sector approximations from [7], or inequalities for Gaussian distributions [25], can be used instead of stochastic sampling.

C. Progressive Introduction of Constraints

Based on these reformulations, we can rewrite the optimization problem as

$$\begin{aligned}&\min_{\underline{\tilde{r}}} \tilde{J}\\ &\text{s. t. } \frac{1}{s} \sum_{j=1}^{s} \mathbf{1}_{\mathcal{P}}(\underline{d}_j) \geq 1 - \alpha \ .\end{aligned}$$

Numerical algorithms for solving this optimization problem (e.g., the interior point method [26]) often require a feasible initial solution, which is nontrivial to obtain, in general. While

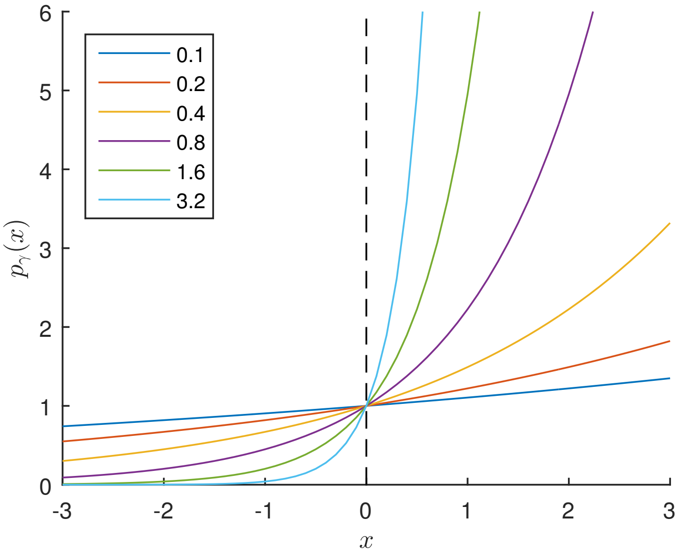

Fig. 2: Value of the penalty term for the constraint $x \leq 0$ depending on the scaling parameter γ.

there are also methods for infeasible starting points [27], those methods typically cannot guarantee that a feasible solution will be found or they take a long time to do so. This is especially problematic when using a sampling-based approach as a small change in the sample locations may not affect the constraint violation at all. In fact, in our experiments as discussed in Sec. IV, we never managed to find a feasible solution using a method for an infeasible starting point. Therefore, we propose to progressively introduce the constraints into the optimization problem. For this purpose, we define a penalty term that is added to the cost function. The proposed penalty term is based on an exponential barrier function rather than the commonly used logarithmic barrier function because the exponential function is well-defined for arbitrary initial values.

For each sample, i.e., for $1 \leq j \leq s$, we define

$$p_\gamma(\underline{d}_j) = \sum_{i=1}^{g} \exp(\gamma \cdot (\mathbf{F}_{i,:} \cdot \underline{d}_j - \underline{b}_i)) \ ,$$

where $\gamma > 0$ is a scaling factor, and $\mathbf{F}_{i,:}$ refers to the i-th row of $\mathbf{F}$. This term serves as a barrier function and is illustrated in Fig. 2. Now, we define a set $Z \subset \{1, \ldots, s\}$ that contains the indices of the $\lfloor \alpha \cdot s \rfloor$ samples with the largest $p_\gamma(\cdot)$. This allows us to define the new cost function

$$\tilde{J}_\gamma = \tilde{J} + \frac{\gamma}{s} \sum_{j=1, j \notin Z}^{s} p_\gamma(\underline{d}_j) \ ,$$

where violations of the constraints are penalized for all samples except those allowed by the violation probability of the chance constraint.

By representing the constraints as part of the cost function, we can now consider the unconstrained optimization problem

$$\min_{\underline{\tilde{r}}} \tilde{J}_\gamma \ .$$

This problem can be solved numerically using a quasi-newton algorithm, such as the Broyden–Fletcher–Goldfarb–Shanno algorithm (BFGS) [27, Ch. 6]. We initially solve the problem for a small value of γ, say $\gamma = 0.1$. Then, we increase γ gradually to a large value, say $\gamma = 100$. At each step, we use the result from the previous step as our new initial solution. As γ gets larger, on the one hand, violations of the chance constraints get penalized more severely, and on the other hand, samples that are close to the constraint but still within the admissible set $\mathcal{P}$ get penalized less. Mathematically speaking, the function $p_\gamma(\underline{x})$ converges pointwise to zero for all $\underline{x} \in \mathcal{P}^O$ (where $\mathcal{P}^O$ is the interior of $\mathcal{P}$) and to infinity for all $\underline{x} \notin \mathcal{P}$.

Of course, the proposed approximation is conservative. However, as simulations indicate, it is less conservative than the approach from [4]. Furthermore, conservatism can be reduced by increasing the number of samples in the last progression step because in this way, the approximation of the probability mass contained within the admissible region gets tighter.

After the progression has finished, it is possible to evaluate the expected constraint satisfaction using the sample approximation $\frac{1}{s}\sum_{j=1}^{s} \mathbf{1}_{\mathcal{P}}(\underline{d}_j)$. If this value is larger than $1-\alpha$, we can deduce that the progressive algorithm has succeeded in finding a feasible solution.

IV. Evaluation

We evaluated the proposed approach in a simulation with planning horizon $K = 50$. For this purpose, we consider a system with parameters

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{R} = 1 \ .$$

The chance constraints are given by

$$\mathbf{F} = \operatorname{diag}(\mathbf{F}_1, \ldots, \mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_2) \ ,$$
$$\underline{b} = [\underline{b}_1^T, \ldots, \underline{b}_1^T, \underline{b}_2^T, \ldots, \underline{b}_2^T]^T \ ,$$

where the parameters $\mathbf{F}_1 = \mathbf{F}_2 = [-1, 0]$, $\underline{b}_1 = -1, \underline{b}_2 = -2$ are repeated $K/2$ times each. Thus, we have the constraint $x_1 \geq 1$ during the first half of the planning horizon and $x_1 \geq 2$ during the second half of the planning horizon.

The initial state of the system is given by $\hat{\underline{x}}_0 = [3, 0]^T$ and the covariance of the system noise w is chosen to

$$\mathbf{\Sigma}_k^w = \begin{bmatrix} 0.1^2 & 0 \\ 0 & 0.05^2 \end{bmatrix} \ .$$

For the stochastic sampling, we use $s = 250$ samples to achieve a good coverage of the probability mass. The maximum allowed constraint violation probability was set to $\alpha = 0.1$.

We compare the proposed controller to the controller presented by van Hessem and Bosgra [4]. For this purpose, we simulated a total of 100 runs with 50 time steps each using both controllers. The results are depicted in Fig. 3.

It can be seen that both controllers manage to satisfy the chance constraints. However, the proposed controller is much less conservative, it gets much closer to the constraint and even violates it on occasion. The signed distance from the constraint is smaller for the proposed controller (mean 0.491, median 0.435) than for van Hessem's controller (mean 0.824, median 0.785). This results in a reduced cost for the proposed controller (mean 231.2, median 230.5) compared to van Hessem's controller (mean 300.8, median 299.9). The resulting cost is also visualized in Fig. 4.

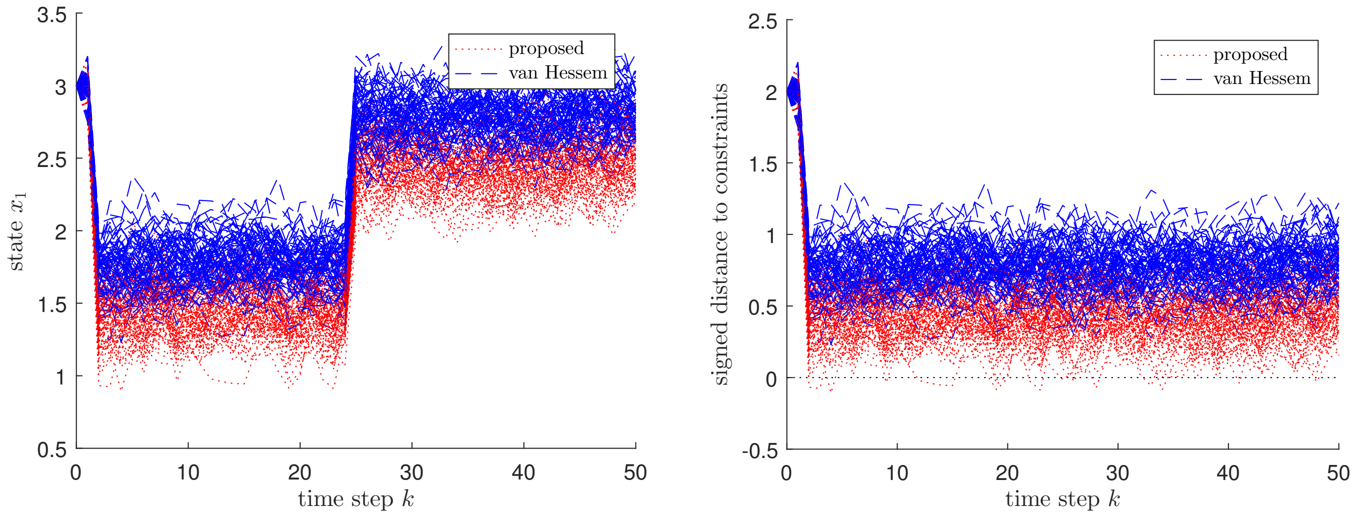

Fig. 3: Example runs of the proposed controller and van Hessem's controller. On the left we show the evolution of x_1 and on the right we illustrate the signed distance from the constraint in each time step. Negative distances indicate constraint violations. It can be seen that the proposed controller is much less conservative and occasionally violates the constraints as it is supposed to.

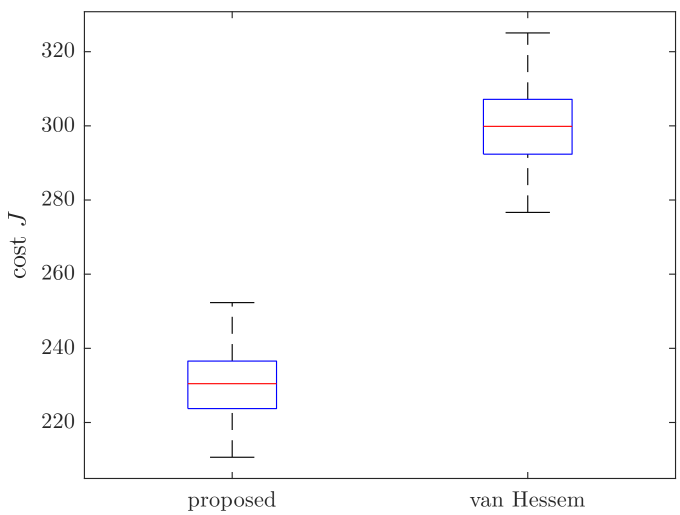

Fig. 4: Cost achieved by both controllers.

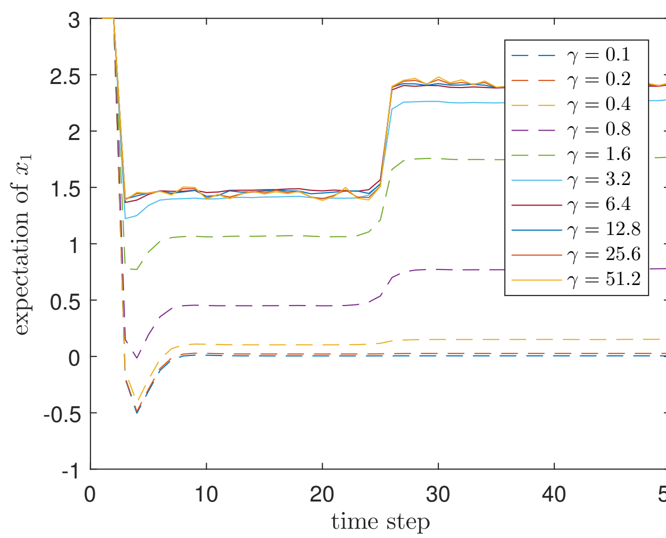

Fig. 5: Influence of the progression parameter γ on the expected trajectory in the first dimension.

In order to illustrate the progression, we show the expected trajectory in the first dimension depending on the parameter γ in Fig. 5. It can be seen that for small γ, the constraints are not taken into account and the controller stabilizes the system towards zero. When γ is increased, the trajectory gradually changes to fulfill the constraints. At some point, further increasing γ does not significantly change the result anymore and the progression can be terminated.

V. Conclusion

In this paper, we have presented a novel approach for closed-loop chance-constrained control based on a progressive introduction of the constraints into the control problem. One of the key advantages of the proposed method is that it is not necessary to know a feasible initial solution of the constrained control problem. Also, in a simulative evaluation the proposed controller showed good performance in comparison to the method proposed by van Hessem et al.

Future work may include a generalization of the presented approach to a system with measurement feedback rather than state feedback, or to a system with elliptic constraints rather than linear constraints. Furthermore, explicitly optimizing for the regulator gains $\mathbf{L}_k$ would allow shaping the covariance such that conservatism could be reduced even further.

References

[1] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained Model Predictive Control: Stability and Optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[2] A. Nemirovski, "On Safe Tractable Approximations of Chance Constraints," *European Journal of Operational Research*, vol. 219, no. 3, pp. 707–718, 2012.

[3] D. van Hessem, C. Scherer, and O. Bosgra, "LMI-based Closed-loop Economic Optimization of Stochastic Process Operation Under State

and Input Constraints," in *Proceedings of the 40th IEEE Conference on Decision and Control*, vol. 5, 2001, pp. 4228–4233.

[4] D. van Hessem and O. Bosgra, "A Conic Reformulation of Model Predictive Control Including Bounded and Stochastic Disturbances Under State and Input Constraints," in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 4, 2002, pp. 4643–4648.

[5] ——, "A Full Solution to the Constrained Stochastic Closed-loop mpc Problem via State and Innovations Feedback and its Receding Horizon Implementation," in *42nd IEEE Conference on Decision and Control*, vol. 1, 2003, pp. 929–934.

[6] M. Dolgov, G. Kurz, and U. D. Hanebeck, "Chance-constrained Model Predictive Control based on Box Approximations," in *Proceedings of the 54th IEEE Conference on Decision and Control (CDC 2015)*, Osaka, Japan, Dec. 2015.

[7] U. D. Hanebeck and M. Dolgov, "Adaptive Lower Bounds for Gaussian Measures of Polytopes," in *Proceedings of the 18th International Conference on Information Fusion (Fusion 2015)*, Washington D. C., USA, Jul. 2015.

[8] A. Nemirovski and A. Shapiro, "Convex Approximations Of Chance Constrained Programs," *SIAM Journal on Optimization*, vol. 17, no. 4, pp. 969–996, 2006.

[9] L. Blackmore and M. Ono, "Convex Chance Constrained Predictive Control Without Sampling," in *Proceedings of the AIAA Guidance, Navigation and Control Conference*, 2009, pp. 7–21.

[10] E. Cinquemani, M. Agarwal, D. Chatterjee, and J. Lygeros, "Convexity and Convex Approximations of Discrete-time Stochastic Control Problems With Constraints," *Automatica*, vol. 47, no. 9, pp. 2082–2087, 2011.

[11] G. C. Calafiore and M. C. Campi, "The Scenario Approach to Robust Control Design," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 742–753, 2006.

[12] A. Nemirovski and A. Shapiro, "Scenario Approximations of Chance Constraints," in *Probabilistic and Randomized Methods for Design Under Uncertainty*. Springer, 2006, pp. 3–47.

[13] L. Blackmore, M. Ono, A. Bektassov, and B. Williams, "A Probabilistic Particle-Control Approximation of Chance-Constrained Stochastic Predictive Control," *IEEE Transactions on Robotics*, vol. 26, no. 3, pp. 502–517, 2010.

[14] M. Prandini, S. Garatti, and J. Lygeros, "A Randomized Approach to Stochastic Model Predictive Control," in *2012 IEEE 51st Annual Conference on Decision and Control (CDC)*. IEEE, 2012, pp. 7315–7320.

[15] X. Zhang, S. Grammatico, G. Schildbach, P. Goulart, and J. Lygeros, "On The Sample Size of Randomized MPC for Chance-constrained Systems with Application to Building Climate Control," in *2014 European Control Conference (ECC)*, 2014, pp. 478–483.

[16] F. Oldewurtel, C. N. Jones, and M. Morari, "A Tractable Approximation of Chance Constrained Stochastic MPC based on Affine Disturbance Feedback," in *47th IEEE Conference on Decision and Control, 2008. CDC 2008.* IEEE, 2008, pp. 4731–4736.

[17] M. Athans, "The Role and Use of the Stochastic Linear-Quadratic-Gaussian Problem in Control System Design," *IEEE Transactions on Automatic Control*, vol. 16, no. 6, pp. 529–552, 1971.

[18] U. D. Hanebeck, "PGF 42: Progressive Gaussian Filtering with a Twist," in *Proceedings of the 16th International Conference on Information Fusion (Fusion 2013)*, Istanbul, Turkey, Jul. 2013.

[19] J. Steinbring and U. D. Hanebeck, "Progressive Gaussian Filtering Using Explicit Likelihoods," in *Proceedings of the 17th International Conference on Information Fusion (Fusion 2014)*, Salamanca, Spain, Jul. 2014.

[20] G. Kurz, I. Gilitschenski, and U. D. Hanebeck, "Nonlinear Measurement Update for Estimation of Angular Systems Based on Circular Distributions," in *Proceedings of the 2014 American Control Conference (ACC 2014)*, Portland, Oregon, USA, Jun. 2014.

[21] ——, "Recursive Bayesian Filtering in Circular State Spaces," *arXiv preprint: Systems and Control (cs.SY)*, Jan. 2015. [Online]. Available: http://arxiv.org/abs/1501.05151

[22] D. Lyons, J.-P. Calliess, and U. D. Hanebeck, "Chance Constrained Model Predictive Control for Multi-Agent Systems with Coupling Constraints," in *Proceedings of the 2012 American Control Conference (ACC 2012)*, Montréal, Canada, Jun. 2012.

[23] G. Schildbach and M. Morari, "Scenario MPC for Linear Time-varying Systems with Individual Chance Constraints," in *American Control Conference (ACC), 2015*, 2015, pp. 415–421.

[24] L. Devroye, *Non-Uniform Random Variate Generation*, 1st ed. New York: Springer, 1986.

[25] P. Hokayem, D. Chatterjee, and J. Lygeros, "Chance-constrained LQG with bounded control policies," in *2013 IEEE 52nd Annual Conference on Decision and Control (CDC)*. IEEE, 2013, pp. 2471–2476.

[26] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY, USA: Cambridge University Press, 2004.

[27] J. Nocedal and S. Wright, *Numerical Optimization*, 2nd ed., ser. Springer Series in Operations Research and Financial Engineering. Springer Science & Business Media, 2006.