

A study on event triggering criteria for estimation

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Abstract—To reduce the amount of data transfer in networked systems measurements are usually taken only when an event occurs rather than periodically in time. However, a fundamental assessment on the response of estimation algorithms receiving event sampled measurements is not available. This research presents such an analysis when new measurements are sampled at well-designed events and sent to a Luenberger observer. Conditions are then derived under which the estimation error is bounded, followed by an assessment of two event sampling strategies when the estimator encounters two different types of disturbances: an impulse and a step disturbance. The sampling strategies are compared via four performance measures, such as estimation-error and communication resources. The result is a clear insight of the estimation response in an event-based setup.

I. INTRODUCTION

Event-based state estimation is an emerging alternative to classical, time-periodic estimators with many relevant applications in networked systems. In contrast to sampling periodically in time, event-based estimators employ an *event triggering* criteria for taking new measurement samples at instants of well designed events. Three examples of event sampling are “Send-on-Delta” [1], “Integral Sampling” [2] and “Matched Sampling” [3], of which two are illustrated in Figure 1. Resource limitations of networked systems in communication and energy are among the main motivations for pursuing event-based estimation, since the event sampling strategy employed aims to reduce the number of measurements exchanged. As such, in the active research area of sensor management (see [4]), event based sampling is a particular sensor scheduling solution related to flexible time-triggered sampling approaches such as [5]. The added value of event sampling is best noticed in networked systems with (simplistic) sensors capturing continuous physical phenomena, e.g., temperature and position sensors. Notice further that employing event sampling will result in some extra computational power at the sensor, so to reduce communication demands.

Control theory was one of the first areas to study event based sampling. See, for example, the event-based control solutions presented in [1], [6]–[8]. More recently, similar studies arose in the estimation area as well, where several solutions for event-based estimation have been presented. These solutions mainly focused on finding a suitable approach for processing the event sampled measurements. For example, by conducting an a-periodic state estimator as proposed in [9] or by employing an estimator specifically designed for event sampling strategies, e.g., the estimators proposed in [10], [11].

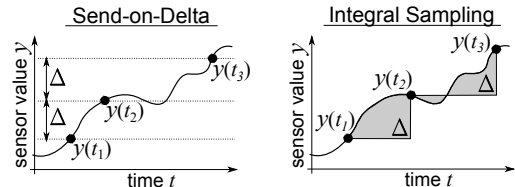


Fig. 1. Two event sampling strategies resulting in different sample instants. One called Send-on-Delta where a new measurement $y(t_e)$ is taken when the distance with respect to the preceding measurement $y(t_{e-1})$ is larger than some scalar Δ , and one called Integral Sampling where the new sample $y(t_e)$ is triggered when the integral $\int_{t_{e-1}}^t \|y(t_{e-1}) - y(v)\| dv$ exceeds the scalar Δ .

Rather than designing a new estimation method, the goal of this paper is to analyze the effects of introducing an event sampling approach for state estimation. To keep focus on the effects instead of optimal estimation results, a Luenberger observer (see [12]) is employed for estimating the state of a scalar process. Important aspects such as the asymptotic behavior of estimation error are analyzed for two event sampling strategies. The remaining analysis focuses on the two fundamental disturbances in the process noise: an impulse disturbance and a step disturbance. The performance of the Luenberger observer in case of these types of disturbances is compared for the two event sampling strategies considered. Important performance measures in this comparison are the number of events triggered, the estimation error at the last event instant (if it exists) and the asymptotic value of the estimation error. In addition, a case study presents an extended event sampling strategy taking the utility of a new measurement sample into account when triggering the next event. The presented study provides insight in the behavior of estimation results, when sensors employ an event sampling strategy.

II. PRELIMINARIES

\mathbb{R} , \mathbb{R}_+ , \mathbb{Z} and \mathbb{Z}_+ define the set of real numbers, non-negative real numbers, integer numbers and non-negative integer numbers, respectively, while $\mathbb{Z}_{\mathbb{P}} := \mathbb{Z} \cap \mathbb{P}$ for some $\mathbb{P} \subset \mathbb{R}$. For a time-varying signal $x(t) \in \mathbb{R}^n$, let $x(t_e)$ denote the value of x at the e -th sampling instant $t_e \in \mathbb{R}_+$ and let $x(t > t_1) = c$ denote that $x(t) = c$ for all $t > t_1$ with similar notations for \geq , $<$ and \leq . Given a set $\mathbb{P} \subset \mathbb{R}$, then $\min \mathbb{P}$ represents the largest possible value such that $\min \mathbb{P} \leq c$, for all $c \in \mathbb{P}$. The floor round-off function $\lfloor x \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ is defined as $\lfloor x \rfloor := \{n \in \mathbb{Z} | n \leq x, n + 1 > x\}$, while the Dirac-function for some value $x \in \mathbb{R}$ is denoted as $\delta(x) : \mathbb{R} \rightarrow \{0, 1\}$.

III. EVENT SAMPLING FOR ESTIMATION

A. System setup

Consider a continuous-time process modeled in state-space, where $x \in \mathbb{R}$ denotes the state vector, $w \in \mathbb{R}$ denotes the process noise and $y \in \mathbb{R}$ denotes the measurement, i.e.,

$$\begin{aligned}\dot{x}(t) &= ax(t) + w(t), & a < 0, \\ y(t) &= cx(t), & c \in \mathbb{R}.\end{aligned}\quad (1)$$

The discrete-time form of the above process model, where $a_\tau := e^{a\tau}$ and $q_\tau(t + \tau) := \int_0^\tau e^{a(\tau-v)}w(t+v)dv$, yields

$$\begin{aligned}x(t + \tau) &= a_\tau x(t) + q_\tau(t + \tau), & \tau \in \mathbb{R}_+, \\ y(t) &= cx(t).\end{aligned}\quad (2)$$

The sensor employs an event strategy for triggering the \mathbf{e} -th measurement sample denoted as $y(t_{\mathbf{e}})$ used for state estimation. Let us introduce $\mathbb{T}_{\mathbf{e}} \subset \mathbb{R}_+$ as the collection of all event instants, i.e., $\mathbb{T}_{\mathbf{e}} := \{t_{\mathbf{e}} \mid \mathbf{e} \in \mathbb{Z}_+\}$. Yet, an estimated value of the state, which is denoted by $\hat{x}(t)$, should not be limited to the instants of an event $t \in \mathbb{T}_{\mathbf{e}}$ but be available at an time $t \in \mathbb{R}$.

The main focus of this article is to study the effects of event sampling on estimation results. More details introducing specific topics of this study will be presented in the next section. For now, it is important to note that the state estimation approach is less relevant for this study than the methods used for event sampling. For that reason a Luenberger observer is employed to estimate the state, which is similar to a Kalman filter when the ‘‘Kalman gain’’ is a constant design parameter (see [12]). Estimation results of this observer are updated with new measurement values at (a-periodic) events $t_{\mathbf{e}}$, when $y(t_{\mathbf{e}})$ is received. In between two consecutive events $t_{\mathbf{e}-1}$ and $t_{\mathbf{e}}$ the state is predicted from the last event $t_{\mathbf{e}-1}$. Hence, estimation results \hat{x} of the Luenberger observer, for some Luenberger gain $\kappa \in \mathbb{R}$, sampling time $\tau_{\mathbf{e}} := t_{\mathbf{e}} - t_{\mathbf{e}-1}$ and unbiased process noise $w(t)$, are characterized as follows:

$$\begin{cases} \hat{x}(t_{\mathbf{e}}) = (1 - \kappa c)a_{\tau_{\mathbf{e}}}\hat{x}(t_{\mathbf{e}-1}) + \kappa y(t_{\mathbf{e}}), \\ \hat{x}(t) = a_{t-t_{\mathbf{e}}}\hat{x}(t_{\mathbf{e}}) \end{cases} \quad \wedge t_{\mathbf{e}} < t < t_{\mathbf{e}+1} \quad (3)$$

The evolution of the true $x(t)$ after an event instant $t_{\mathbf{e}}$ is derived from (2), and yields

$$x(t) = a_{t-t_{\mathbf{e}}}x(t_{\mathbf{e}}) + q_{t-t_{\mathbf{e}}}(t_{\mathbf{e}}) \quad \wedge t > t_{\mathbf{e}}. \quad (4)$$

A graphical representation of the system is depicted in Figure 2. Therein, the sensor value is continuously monitored for events and sampled measurements are sent to the Luenberger observer in case of an event. The event triggering criteria can make use of a local Luenberger observer situated at the sensor.

B. Problem formulation

An important aspect for any observer is its estimation error, which will be denoted as $\eta(t) := x(t) - \hat{x}(t)$. The evolution of this error for the considered Luenberger observer has been derived from (3) and (4) and yields

$$\begin{cases} \eta(t_{\mathbf{e}}) = (1 - \kappa c)(a_{\tau_{\mathbf{e}}}\eta(t_{\mathbf{e}-1}) + q_{\tau_{\mathbf{e}}}(t_{\mathbf{e}})), \\ \eta(t) = a_{t-t_{\mathbf{e}}}\eta(t_{\mathbf{e}}) + q_{t-t_{\mathbf{e}}}(t) \end{cases} \quad \wedge t_{\mathbf{e}} < t < t_{\mathbf{e}+1}. \quad (5)$$

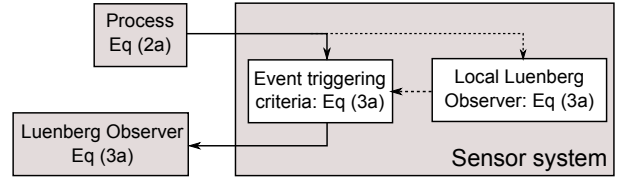


Fig. 2. The considered set-up for event-based state estimation.

A ‘‘stable’’ observer implies that the estimation error $\eta(t_{\mathbf{e}})$ is bounded for all $\mathbf{e} \in \mathbb{Z}_+$. One can then derive from (5) that stability of the considered Luenberger observer will require $0 < (1 - \kappa c)a_{\tau_{\mathbf{e}}} < 1$, for all $\tau_{\mathbf{e}} \in \mathbb{R}_+$. Starting from $a < 0$ it follows that $0 < a_{\tau_{\mathbf{e}}} < 1$, for all $\tau_{\mathbf{e}} \in \mathbb{R}_+$. Thus, $0 < \kappa c < 1$ is a necessary condition for obtaining stability.

The evolution of the estimation error introduced in (5) is the starting point for studying the effects of event-based sampling in state estimation. The study starts with a stability analysis of the Luenberger observer agreeing to $0 < \kappa c < 1$. More precisely, stability is studied as a dependency on the disturbance $w(t)$ and on the event sampling strategy employed.

In addition, a comparison study of the estimator’s performance is made for two different event sampling strategies. One strategy is purely based on sensory data and does not require a local Luenberger observer, while the other sampling strategy is allowed to exploit results of this local observer. The comparison focuses on the estimation error in time and on the required communication resources when the process is exited at $t_d \in \mathbb{R}_+$ with one of the following two disturbances:

- D1: An impulse on the process noise, i.e., $x(0) - \hat{x}(0) = 0$ and $w(t) = \varepsilon\delta(t - t_d)$, for some $t_d, \varepsilon \in \mathbb{R}_+$;
- D2: A step on the process noise, i.e., $x(0) - \hat{x}(0) = 0$, $w(t < t_d) = 0$ and $w(t \geq t_d) = \varepsilon$, for some $t_d, \varepsilon \in \mathbb{R}_+$.

For a fair comparison, the error behavior is assessed with four different performance measures $M1$ until $M4$.

- M1: $\alpha \in \mathbb{Z}_+$ is the number of samples \mathbf{e} generated after t_d , i.e., α is the cardinality of $\{\mathbf{e} \in \mathbb{Z}_+ \mid t_{\mathbf{e}} > t_d\}$;
- M2: $t_\alpha - t_d$ is the time until the last event was triggered;
- M3: $\eta(t_\alpha) \in \mathbb{R}$ is the estimation error at the last event t_α .
- M4: $\eta(\infty) \in \mathbb{R}$ is the asymptotic estimation error.

The presented studies on estimation error start with a stability analysis, followed by the introduction of two typical event sampling strategies. After that, the analysis continues with the two fundamental disturbances and concludes with a discussion on triggering new events depending on utility.

IV. ASYMPTOTIC ANALYSIS OF THE ESTIMATION ERROR

Stability of the event-based Luenberger observer is studied via a continuous-time description of its estimation error in (5).

When considering continuous time, the evolution of the estimation error $\eta(t)$ as presented in (5) shows a switching character, where switching takes place at the events $t \in \mathbb{T}_{\mathbf{e}}$. More precisely, the error characterization in (5) can be described as the continuous-time outcome of a sequence of subsystems, each denoted as $\Sigma_{\mathbf{e}}$. A subsystem $\Sigma_{\mathbf{e}}$ is active

for all time $t \in [t_e, t_{e+1})$ and starts with an initial value $\eta(t_e)$ dependent on the results of previous subsystem Σ_{e-1} , i.e., $\eta(t_e) = (1 - \kappa c) \lim_{t \uparrow t_e} \eta(t)$. To derive the evolution of $\eta(t)$ within each subsystem Σ_e , let us rewrite the discrete-time form of $\eta(t)$ in (5), i.e., $\eta(t) = a_{t-t_e} \eta(t_e) + q_{t-t_e}(t)$, into its continuous form, i.e., $\dot{\eta}_x(t) = a\eta(t) + w(t)$. Then, the evolution of $\eta(t)$ within each subsystem is defined as follows:

$$\Sigma_e : \quad \dot{\eta}_x(t) = a\eta(t) + w(t), \quad \forall t \geq t_e, \\ \text{and } \eta(t_e) := (1 - \kappa c) \lim_{t \uparrow t_e} \eta(t).$$

Though each subsystem Σ_e is characterized for $t \rightarrow \infty$, note that it could be taken over by the next subsystem Σ_{e+1} at $t = t_{e+1}$ (if triggered). The above characterization of $\eta(t)$ via a switched system is done for two reasons: 1. in case t_e is the last sample instant, then Σ_e will run for $t \rightarrow \infty$ and; 2. stability of the total system can be decomposed into stability per subsystem Σ_e . Hence, to prove under which conditions the evolution of $\eta(t)$ in (5) is stable, i.e., converging, let us first prove stability of each subsystem Σ_e . *Lyapunov stability* of Σ_e , for all $e \in \mathbb{Z}_+$, requires that $w(t) = 0$, for all $t \in \mathbb{R}_+$. Since $w(t)$ is unknown process noise, Lyapunov stability can never be guaranteed. Instead, conditions for *input-to-state stability* (ISS), i.e. $\eta(t) < \infty$ for all $t \in \mathbb{R}_+$, are derived.

Lemma IV.1 *Let $\eta(t_e) < \infty$ be given for all $t_e \in \mathbb{R}_+$ and let $a < 0$. Then, each subsystem Σ_e in (6) is ISS, for all $e \in \mathbb{Z}_+$.*

The proof of this lemma is found in (Section 4.9, [13]). Let us use this result to present a statement regarding the stability of the considered Luenberger observer.

Lemma IV.2 *Let $\eta(0) < \infty$, $0 < \kappa c < 1$ and $a < 0$. Then, the sequence of subsystems Σ_e described in (6) and thus the evolution of $\eta(t)$ in (5) satisfies the conditions for ISS.*

Proof: Lemma IV.1 gives that each subsystem Σ_e is ISS, if $\eta(t_e) < \infty$, for all $e \in \mathbb{Z}_+$. This further implies that the sequence of subsystems, and thus the evolution of $\eta(t)$ as characterized in (5), is ISS if $\eta(t_e) < \infty$ holds for all $e \in \mathbb{Z}_+$. The value of $\eta(t_e)$ is defined in (6), i.e. $\eta(t_e) = (1 - \kappa c) \lim_{t \uparrow t_e} \eta(t)$. Applying $0 < (1 - \kappa c) < 1$, gives that $\eta(t_e) < \lim_{t \uparrow t_e} \eta(t)$. Notice that $\{\lim_{t \uparrow t_e} t\} \in (t_{e-1}, t_e)$ and thus $\lim_{t \uparrow t_e} \eta(t)$ is the result of Σ_{e-1} at t_e , which is ISS if $\eta(t_{e-1}) < \infty$. Hence, $\eta(t_e) < \infty$ holds for all $e \in \mathbb{Z}_+$ if $\eta(t_{e-1}) < \infty$. Starting from $\eta(t_0) < \infty$ in combination with the result of Lemma IV.1, it thus follows that $\eta(t_e) < \infty$, for all $e \in \mathbb{Z}_+$, and thus $\eta(t)$ in (5) is ISS. ■

This completes the general analysis of event sampling for the Luenberger observer. In continuation of this analysis, two specific event triggering criteria are assessed when facing two fundamental disturbances.

V. TWO GENERAL EVENT SAMPLING STRATEGIES

1) *Send-on-Delta*: The event sampling strategy ‘‘Send-on-Delta’’ (SoD), as proposed by [1], triggers a new event sample

instant t_{e+1} in case $|y(t) - y(t_e)|$ is larger then some predefined value $\Delta \in \mathbb{R}_+$. Hence, the triggering criteria of the next event instant t_{e+1} for SoD, yields

$$t_{e+1} := \min\{t > t_e \mid |y(t) - y(t_e)| \geq \Delta\}.$$

Since $y(t) = c(a_{t-t_e}x(t_e) + q_{t-t_e}(t))$ and $y(t_e) = cx(t_e)$, the triggering condition of SoD results in the following property:

$$|c(a_{t-t_e} - 1)x(t_e) + cq_{t-t_e}(t)| < \Delta, \quad \forall t_e < t \leq t_{e+1}. \quad (7)$$

2) *Predicted sampling*: The event sampling strategy ‘‘Predicted Sampling’’ (PS) triggers a new event sample instant $t = t_{e+1}$ in case $|y(t) - \hat{y}(t)|$ is larger then some predefined value $\Delta \in \mathbb{R}_+$. Here, $\hat{y}(t)$ is the predicted measurement of the local Luenberger observer. Hence, the event instant t_{e+1} according to PS yields,

$$t_{e+1} := \min\{t > t_e \mid |y(t) - \hat{y}(t)| \geq \Delta\}. \quad (8)$$

Notice that SoD does not result in a bound on the estimation error, as shown in (7). In contrast to that, the triggering condition of PS was derived by starting from a bound on the (modeled) estimation error $\eta(t)$, i.e.,

$$|c\eta(t)| < \Delta \quad \forall t \in \mathbb{R}_+, \\ \Rightarrow |ca_{t-t_e}\eta(t_e) + cq_{t-t_e}(t)| < \Delta \quad \forall t_e < t \leq t_{e+1}, \\ \Rightarrow |c(a_{t-t_e}x(t_e) + q_{t-t_e}(t)) - ca_{t-t_e}\hat{x}(t_e)| < \Delta \quad \forall t_e < t \leq t_{e+1}.$$

Substituting $y(t) = c(a_{t-t_e}x(t_e) + q_{t-t_e}(t))$ and $\hat{y}(t) = ca_{t-t_e}\hat{x}(t_e)$, for all $t > t_e$, recovers the triggering condition (8).

VI. ESTIMATION ERROR UNDER DISTURBANCES

This section on the assessment of SoD and PS in case of disturbances will use $\kappa_c := (1 - \kappa c)$ for clarity of expression.

A. Impulse disturbance

This section analyzes the response of the event-based Luenberger observer in case of an impulse disturbance after starting at its equilibrium, i.e., $x(0) = 0$, $\hat{x}(0) = 0$ and $w(t) = \varepsilon\delta(t - t_d)$. Note that the process’ state and the observer’s estimated state will remain zero until the disturbance at t_d takes place, which further implies that the observer will trigger no events before the disturbance. At t_d an impulse is injected into the process via its process noise, which results in the following behavior on the process’ state after the disturbance, i.e.,

$$x(t \geq t_d) = \varepsilon e^{a(t-t_d)} \quad \text{and} \quad y(t) = cx(t). \quad (9)$$

Since the process and the observer are initialized with $x(0) = 0$ and $\hat{x}(0) = 0$, the estimation error $\eta(t)$ derived in (5) remains zero until the impulse is injected. No event strategy is able to trigger an event at the exact same instant as the disturbance and thus $\eta(t_d) = \varepsilon$. After that, the discrete-time process noise $q_{t-t_d}(t) = 0$ remains zero for all $t > t_d$. Hence, when the disturbance is detectable ($\Delta < c\varepsilon$) both SoD and PS trigger a first event directly after t_d , i.e., $t_1 = \lim_{t \uparrow t_d} t$, at which

$$\eta(t_1) = \kappa_c a_{t_1-t_d} \eta(t_d) = \kappa_c \varepsilon. \quad (10)$$

After this event, the evolution of $\eta(t)$ for all $t > t_1$, yields

$$\begin{cases} \eta(t_{e+1}) = \kappa_c a_{\tau_e} \eta(t_e), & \forall t_e > t_d, \\ \eta(t) = a_{t-t_e} \eta(t_e) & \forall t_e < t < t_{e+1}. \end{cases} \quad (11)$$

The error analysis starts by summarizing the performance measures $M1$ until $M4$ for both sampling strategies in Table I. Their derivations are presented in Appendices A and B. In addition, Figure 3 depicts the results of a simulated impulse, where $a = -1$, $\varepsilon = 1$, $c = 1$, $\kappa = 0.5$ and $\Delta = 0.1$.

	Send-on-Delta	Predicted Sampling
α	$\lfloor c\varepsilon\Delta^{-1} \rfloor$	$\min\{n \in \mathbb{Z}_{>1} \mid c(\kappa_c)^n < \Delta\}$
$t_\alpha - t_d$	$a^{-1} \ln(1 - (\alpha - 1)\Delta(c\varepsilon)^{-1})$	$\lim_{\tau \downarrow 0} \tau$
$\eta(t_\alpha)$	$(\kappa_c)^\alpha \varepsilon (1 - (\alpha - 1)\Delta(c\varepsilon)^{-1})$	$(\kappa_c)^\alpha$
$\eta(\infty)$	0	0

TABLE I
MEASURES $M1$ UNTIL $M4$ FOR AN IMPULSE DISTURBANCE.

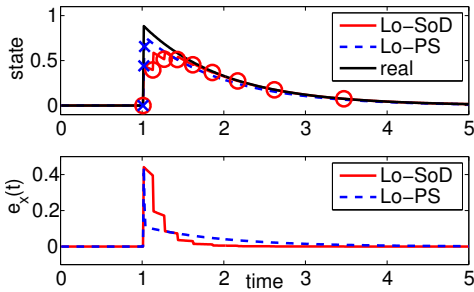


Fig. 3. A simulation example of the event-based Luenberger observer (Lo) in case of an impulse disturbance at $t_d = 1$. The event instants of SoD are denoted with “o” while the event instants of PS are denoted with “x”.

Figure 3 shows that PS has all of its events co-located in time directly after the disturbance. This is to make sure that the estimation error decreases to a value Δc^{-1} as fast as possible (see Section V.2). As a result, the error of the Luenberger observer has a faster decay when combined with PS than with SoD. However, after some time the events triggered with SoD result in an estimation error that remains smaller compared to error with PS, though this aspect depends on Δ and c .

B. Step disturbance

This section analyzes the response of the event-based Luenberger observer in case of a step disturbance after starting at its equilibrium, i.e., $x(0) = 0$, $\hat{x}(0) = 0$, $w(t < t_d) = 0$ and $w(t \geq t_d) = \varepsilon$. Note that the process’s state and the observer’s estimated state remain zero until the disturbance at t_d takes place, which further implies that the observer will trigger no events before the disturbance. At t_d a step is injected into the process via its process noise, due to which the behavior of the process’ state after the disturbance, yields

$$x(t \geq t_d) = \frac{\varepsilon}{a} (e^{a(t-t_d)} - 1) \quad \text{and} \quad y(t) = cx(t). \quad (12)$$

The above characterization follows from the evolution of $w(t)$ in its discrete-time form presented in Section III, $q_\tau(t + \tau) =$

$\int_0^\tau e^{a(\tau-v)} w(t+v) dv = \varepsilon \int_0^\tau e^{a(\tau-v)} dv$ for all $t \geq t_d$ and thus

$$\begin{aligned} q_\tau(t + \tau) &= \frac{\varepsilon}{a} (e^{a\tau} - 1) \\ &= \frac{\varepsilon}{a} (a_\tau - 1) \quad \forall t \geq t_d. \end{aligned} \quad (13)$$

The characterization in (12) implies that $x(t_d) = 0$ and thus $\eta(t) = 0$, for all $t \leq t_d$. An expression of estimation error after the disturbance at t_d and just before the first event triggered is derived, next. Starting from $\eta(t \geq t_d) = x(t \geq t_d) - \hat{x}(t \geq t_d)$ and substituting $x(t \geq t_d)$ of (12) and $\hat{x}(t \geq t_d) = 0$ for any $t < t_1$ results in the following error characterization until the first event t_1 takes place:

$$\begin{cases} \eta(t) = \frac{\varepsilon}{a} (a_{t-t_d} - 1), & \forall t_d \leq t < t_1, \\ \eta(t) = \kappa_c \frac{\varepsilon}{a} (a_{t-t_d} - 1), & \text{if } t = t_1. \end{cases}$$

With this result and the characterization of $q_\tau(t + \tau)$ in (13) one can derive the estimation error after the first event, i.e.,

$$\begin{cases} \eta(t_{e+1}) = \kappa_c (a_{\tau_e} \eta(t_e) + \frac{\varepsilon}{a} (a_{\tau_e} - 1)), \\ \eta(t) = a_{t-t_e} \eta(t_e) + \frac{\varepsilon}{a} (a_{t-t_e} - 1) & t_e < t < t_{e+1}. \end{cases} \quad (14)$$

When the disturbance is large enough to be detected, i.e., $\Delta < c\varepsilon$, then both SoD and PS will trigger their first event some time after the disturbance. A detailed analysis starts by deriving the different performance measures $M1$ until $M4$ for both sampling strategies. The derivations are presented in Appendices C and D and a summary is given in Table II. To that extent, let us introduce the functions $\gamma(n, m) := \frac{c+a\Delta n}{c+a\Delta m}$ and $\beta(m) := \frac{\varepsilon\Delta}{c+a\Delta m}$, for some $n, m \in \mathbb{Z}_+$. In addition, Figure 4 depicts the results of a simulated step, where $a = -1$, $\varepsilon = 0.01$, $c = 1$, $\kappa = 0.5$ and $\Delta = 0.1$.

	Send-on-Delta	Predicted Sampling
α	$\lfloor \lfloor \varepsilon c (a\Delta)^{-1} \rfloor \rfloor$	∞
$t_\alpha - t_d$	$a^{-1} \ln(1 + \frac{a\alpha\Delta}{\varepsilon c})$	∞
$\eta(t_\alpha)$	$\kappa_c^\alpha c^{-1} \Delta \gamma(\alpha, 1) \eta(t_1) + \sum_{m=1}^{\alpha-1} \kappa_c^{\alpha-m} \gamma(\alpha, m+1) \beta(m)$	—
$\eta(\infty)$	$-\varepsilon a^{-1}$	$< \Delta c ^{-1}$

TABLE II
MEASURES $M1$ UNTIL $M4$ FOR A STEP DISTURBANCE.

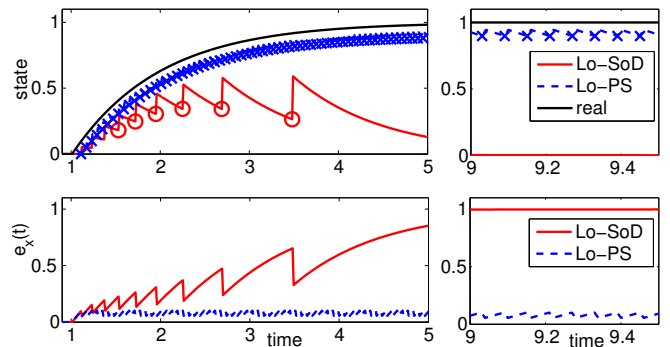


Fig. 4. A simulation example of the event-based Luenberger observer (Lo) in case of a step disturbance at $t_d = 1$. The event instants of SoD are denoted with “o” while the event instants of PS are denoted with “x”.

Two interesting aspects are indicated in Table II and Figure 4. Firstly, the estimation error with SoD has an asymptotic bias of $-\frac{\varepsilon}{a} = 1$. An explanation for this behavior starts by noting that SoD triggers a finite amount of events. After the last event the event-based Luenberger observer will keep on predicting the estimated state $\hat{x}(t)$ and since the observer assumes an unbiased process noise, $\hat{x}(t)$ converges back to zero while the real state $x(t)$ converges to 1 giving $\eta(\infty) = 1$. Event-based estimation with PS does not suffer from such a severe asymptotically biased estimation error but is bounded by 0.1, i.e., $\Delta|c|^{-1}$. It shows that the property of PS resulting in a bounded estimation error $|c\eta(t)| < \Delta$ for all $t \in \mathbb{R}_+$ is satisfied, see Section V.2. The cost of this bound is that an unlimited amount of events are required. Intriguingly, the event instants seem to occur periodically in time, which brings us to the second interesting aspect. After each event t_e , one can derive that $a_{t-t_e} \downarrow 0$ as $t - t_e \rightarrow \infty$, which further implies that $\eta(t)$ converges to $-\frac{\varepsilon}{a} = 1$ after each event t_e (see (14)). Yet, PS will trigger a new event just before $|c\eta(t)| > \Delta$ becomes reality and thus $\eta(t)$ will never converge to $-\frac{\varepsilon}{a}$. This situation for triggering new events will continue until $t \rightarrow \infty$ and results in a constant sampling time between two consecutive events. A detailed account in this sampling time, which is denoted as τ_s , is presented in Appendix D and yields

$$\tau_s = \frac{1}{a} \ln \left(\frac{|\Delta c^{-1}| + \varepsilon a^{-1}}{\kappa_c |\Delta c^{-1}| + \varepsilon a^{-1}} \right).$$

VII. ILLUSTRATIVE CASE FOR TRIGGERING ON UTILITY

The illustrative case presented in this section considers a scalar dynamical system as given in (1), i.e.,

$$\dot{x}(t) = ax(t) + w(t) \quad \text{and} \quad y(t) = cx(t),$$

where $a = -1$ and $c = 1$. The process noise $w(t)$ is characterized with impulse disturbances, given that the time between two impulses is a random value between 0 and 1 seconds, and that the impulse amplitudes are Gaussian distributed (unbiased and a covariance of 1). More precisely, $w(t)$ is given by

$$w(t) = \sum_{n \in \mathbb{Z}_+} \varepsilon_n \delta(t - t_n), \quad \text{where} \quad (15)$$

$$\varepsilon_n \sim \mathcal{N}(0, 1) \quad \text{and} \quad 0 < t_{n+1} - t_n < 1 \quad \forall n \in \mathbb{Z}_+.$$

A Luenberger observer with $\kappa = 0.5$ estimates the state by exploiting new measurements taken with an event sampling strategy, e.g., SoD or PS. The Luenberger observer and sampling strategies are designed for perceiving the following goal:

Track the state $x(t)$, such that any disturbance on $x(t)$ is corrected in the estimated state $\hat{x}(t)$ within 0.1 seconds up to an acceptable error of $|x(t) - \hat{x}(t)| \leq 0.2$.

A. Event triggering with classical SoD or PS

Let us start by designing the threshold Δ of the event strategies SoD and PS, such that they accommodate the above goal. Note that the illustrative case focuses on impulse disturbances, due to which the analysis of Section VI-A holds.

SoD: The design of the threshold Δ for SoD was done by trial and error, which resulted in $\Delta = 0.05$. The reason that a trial-and-error approach was employed is because the estimation error of a Luenberger observer with SoD depends on the amplitude of prior impulses, which vary from one impulse to another and can be unbounded with a very small probability.

PS: To derive a value of Δ , note that it was already pointed out that PS will trigger as many events directly after each other until $|c\eta(t_\alpha)| < \Delta$ is attained. Hence, all events caused by one impulse disturbance will be triggered within 0.1 seconds. Further, when setting $\Delta = 0.2$, one has a guarantee of $|c\eta(t_\alpha)| < 0.2$ satisfying the acceptable estimation error of 0.2.

A simulation was done with the above thresholds for a 'train' of 1000 impulses in line with (15). To assess the results, let us concentrate on two important aspects of the estimator, i.e., its estimation results and the required communication resources for achieving those results. More precisely,

- The estimation error of the Luenberger observer after 0.1 seconds of the n -th impulse, i.e., $\eta(t_n + 0.1)$;
- The time interval between two consecutive events, where only intervals shorter than 0.2 seconds were considered¹.

Figure 5 illustrates a histogram of $|\eta(t_n + 0.1)|$ for a bin-size of 0.1. The figure shows that the SoD strategy is not able to satisfy the observer's goal, i.e., uphold a 0.2 error bound within 0.1 seconds of the impulse disturbance. This aspect is due to the fact that one cannot make any guarantees on the estimation error when SoD is employed as the amplitude of the impulse could become unbounded. For PS, which in theory does guarantee an upper bound on the estimation error, simulations showed that $|\eta(t_n + 0.1)|$ seldom exceeds the bound of 0.2. This is most likely due to occasional situations where two impulses succeed each other within 0.1 seconds (which has a probability of 10%). As such, there is a probability of around 10% that the estimation error with PS exceeds the value of 0.2 after 0.1 seconds from the impulse. A drawback of PS is illustrated in Figure 6 indicating that PS triggered all events directly after each other, which was also expected. This aspect implies that communication resources at the instants of an impulse are enormous, otherwise one cannot send measurements directly after each other. The alternative SoD strategy shows a more even distribution of events, though the total number of events, i.e., the integral of the histogram, is much more compared to PS. One can conclude that neither of the two meets the estimation goal with acceptable communication resources and a different sampling approach is thus desired.

B. Event triggering based on utility

The issue with SoD and PS is that they do not account for any dynamics in the utility of new sensor measurements, i.e., how useful it would be to receive new measurements. Instead,

¹Two consecutive events with an interval larger than 0.2 are assumed to have been caused by *two different* impulses. Since our concern is the performance of the estimator per impulse, such intervals are not considered.

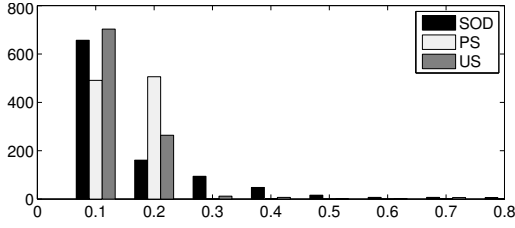


Fig. 5. A histogram of the absolute estimation error at 0.1 seconds after an impulse disturbance for the considered case of 1000 impulses.

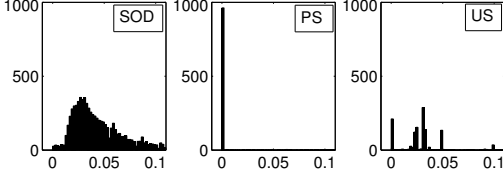


Fig. 6. A histogram of the time interval between two consecutive events that were caused by the same impulse disturbance.

both strategies define a static triggering condition making the importance of new measurements fixed. The example studied here shows that even for a simplistic case of impulse disturbances this importance is not fixed, for example, due to uncertainty on when and how big the next impulse will be. Hence, the importance of new measurement samples depends on current estimation results, i.e., on previous disturbances, and on the characteristics of the currently encountered disturbance: amplitude and timing. Therefore, a dynamic event sampling strategy is developed based on the current importance (or utility) of new measurements.

The dynamic triggering condition proposed here, called Utility Sampling (US), takes the evolution of the sensor value $y(t)$ as expected by the estimator into account and introduces a dynamic threshold $\Delta(t)$ set by the estimator.

Let us start with a characterization of $y(t)$ as expected by the estimator, which is derived from process model in (1) and (2). The model points out that $y(t) = cx(t)$, which further implies that $y(t) = ca_{t-t_e}x(t_e) = e^{a(t-t_e)}cx(t_e) = e^{a(t-t_e)}y(t_e)$. Hence, after the e -th event, i.e., triggering $y(t_e)$, the estimator expects that the measurement will evolve as $y(t) = e^{a(t-t_e)}y(t_e)$, for a given value of a . As such, where SoD triggers a new event whenever $y(t)$ is a Δ -distance away from $y(t_e)$, one can now define that a new event should be triggered whenever $y(t)$ is a Δ -distance away from the expected sensor value $e^{a(t-t_e)}y(t_e)$, thereby trigger a new event whenever $|e^{a(t-t_e)}y(t_e) - y(t)| > \Delta$. Such a triggering condition implies that a new measurement sample is important whenever the current sensor value $y(t)$ is outside a Δ -distance from the expected sensor value $e^{a(t-t_e)}y(t_e)$. The next step is to introduce a dynamic threshold $\Delta(t)$ characterizing the importance of a new measurement in relation to the goal of the estimator. Before deriving this threshold, note that the triggering criteria of the next event (sample) instant t_{e+1} for US, yields

$$t_{e+1} := \min\{t > t_e \mid |y(t) - e^{a(t-t_e)}y(t_e)| \geq \Delta(t)\}.$$

In order to derive a description of $\Delta(t)$, let us recall that the goal is to reduce the estimation error up until 0.2 within 0.1 seconds of the n -th impulse disturbance occurring at a time t_n . The estimation error at any time $t > t_n$ depends on the estimation error initially at the instant of the n -th event $\eta(t_n)$, which can be perceived via $\eta(t_n) = x(t_n) - \hat{x}(t_n) = c^{-1}y(t_n) - \hat{x}(t_n)$. From Section VI-A one can then obtain that the estimation error at the event instants evolves as $\eta(t_{e+1}) = \kappa_c a \tau_e \eta(t_e) \leq \kappa_c \eta(t_e)$. As such, for an initial estimation error of $\eta(t_n) = c^{-1}y(t_n) - \hat{x}(t_n)$, the estimation error at the e -th event triggered after the n -th impulse satisfies $\eta(t_e) \leq \kappa_c^e \eta(t_n)$, which further implies the following inequality

$$\eta(t_e) \leq \kappa_c^e (c^{-1}y(t_n) - \hat{x}(t_n)).$$

The above equation can be used to compute the amount of events e_n^* necessary for reducing the estimation error to a value less than 0.2, i.e., $e_n^* := \{e \in \mathbb{Z}_+ \mid \kappa_c^e (c^{-1}y(t_n) - \hat{x}(t_n)) < 0.2\}$. Communication resources are the least in case these event instants are evenly distributed within the 0.1 seconds after the n -th event. In case the first event will be triggered at t_n for detecting the disturbance, this principle implies that a next event should be triggered every $\tau_n = 0.1/(e_n^* - 1)$ seconds after t_n . In addition, one should also be prepared for new impulses triggered within 0.1 seconds after the n -th impulse. Therefore, the value of Δ adopts the followed characterization:

$$\Delta(t) := \begin{cases} (0.2(t_{e-1} + \tau_n) - t) & \text{if } t_n < t \leq t_n + 1 \\ 0.2 & \text{if } t > t_n \end{cases}$$

The above characterization implies that $\Delta(t)$ turns negative at carefully selected periodic instances in between t_n and $t_n + 0.1$ so to obtain an estimation error lower than 0.2 within 0.1 seconds after the n -th disturbance. Furthermore, the fact that $\Delta \leq 0.2$ also implies that new impulses occurring shortly after the n -th impulses will be detected and trigger an event. The constant threshold $\Delta = 0.2$ for any $t_n + 0.1 < t \leq t_{n+1}$ implies that after the estimator has returned to an equilibrium, i.e., after $t_n + 0.1$, the threshold $\Delta = 0.2$ will imply that new impulses causing the estimation error to increase above the 0.2 will be detected and trigger a next event.

Results of this event sampling strategy on the estimation error at $t_n + 0.1$ and on the sampling time in between two consecutive events are depicted in Figure 5 and Figure 6, respectively. They show that US is an event sampling approach able to keep the estimation error within the 0.2 bounds after 0.1 seconds of the impulse disturbance for almost every impulse. In addition, communication resources are relaxed since the time in between two consecutive events is more distributed within the 0.1 seconds instead of being triggered directly after each other (as is the case with PS).

VIII. DISCUSSION ON EVENT VERSUS PERIODIC

So far, the analysis in this article has been based on a scalar system with specific disturbances. A final comparison for a realistic scenario is presented in this section. The scenario considered is a 2D walk, where the state x consists of position

and speed in a plane and, similarly, acceleration is modeled as 2D Gaussian process noise. Further, the sensor measures the 2D position but is effected by additive Gaussian measurement noise (unbiased and a covariance of 0.01). The Luenberger observer is the selected estimator, where $\kappa = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 \end{pmatrix}^\top$

Figure 7 depicts the results of 1000 simulated walks of 10 seconds each. The three sampling strategies employed were 'Send-on-Delta' (SoD) with a threshold of 0.2, 'Predictive Sampling' (PS) with a threshold of 0.175 and 'Time-Periodic' sampling (TP) with a sampling time of 0.33 seconds. The results depicted are the number of sampling instant (for TP always 31) and the total squared estimation error per simulated walk. The figure shows that PS outperforms the other sampling strategies in both number of samples as well as estimation error. The other two strategies, i.e., SoD and TP, have comparable estimation error results, though TP requires a bit more samples (on average) to obtain its results.

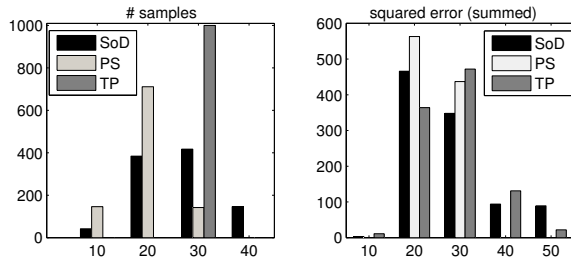


Fig. 7. The 1000 simulations divided in a histogram of the squared estimation error (summed over the entire 10 seconds) and the amount of samples triggered during those 10 seconds.

IX. CONCLUSIONS

The article presents a fundamental assessment on the response of a Luenberger observer when new measurements are triggered in line with an event sampling strategy. Conditions for an asymptotically bounded estimation error of the Luenberger observer were derived. Further, two existing and one novel event sampling strategy were presented, each having a particular characterizing aspect, i.e., sensor history based, sensor prediction based, utility based. Theoretical results were obtained for the two existing sampling strategies by deriving the number of events, their event instances and (asymptotic) estimation errors when the observed process was effected by either an impulse or a step disturbance. A simulation case study revealed that the two existing sampling strategies could not uphold the estimator's goal of achieving particular error bounds within a given time after the disturbance was detected. Therefore, a third (novel) event strategy was developed starting from the utility of new measurement samples for the estimator's goal. This third strategy outperformed both existing strategies in communication resources, while having similar (slightly improved) results in the estimation error.

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APPENDIX

A. Send-on-Delta: Impulse disturbance

For introduction, recall that the first event after the disturbance is triggered at $t_1 = \lim_{t \downarrow t_d} t$ to result in the estimation error $\eta(t_1) = \kappa_c \varepsilon$ (see Section VI-A). The next event of SoD depends on the evolution of $y(t)$ after the disturbance, which follows $y(t) = cx(t) = c\varepsilon e^{a(t-t_d)}$ for all $t \geq t_d$ (see ((9)). Notice that $y(t_1) = c\varepsilon$, after which $y(t)$ will converge to 0. Hence, as soon as $y(t)$ dropped from $c\varepsilon$ to $c\varepsilon - \Delta$, one obtains $|y(t) - y(t_{e=1})| \geq \Delta$ and thus $t_{e=2}$ is triggered. In general, the e -th event is triggered when $y(t) = c\varepsilon - (e-1)\Delta$, i.e.,

$$\begin{aligned} c\varepsilon e^{a(t_e - t_d)} &= c\varepsilon - (e-1)\Delta, \\ \Rightarrow t_e &= t_d + \frac{1}{a} \ln \left(\frac{c\varepsilon - (e-1)\Delta}{c\varepsilon} \right). \end{aligned} \quad (16)$$

Hence, the inter-sampling time τ_e , for all $e \in \mathbb{Z}_+$, yields

$$\tau_e = t_{e+1} - t_e = \frac{1}{a} \ln \left(\frac{c\varepsilon - (e)\Delta}{c\varepsilon - (e-1)\Delta} \right).$$

Then, equation (11) gives that the estimation error at t_{e+1} follows $\eta(t_{e+1}) = \kappa_c a_{\tau_e} \eta(t_e)$, which can be rewritten with the above characterization of τ_e and $a_{\tau_e} = e^{a\tau_e}$ into

$$\begin{aligned} \eta(t_{e+1}) &= \kappa_c \left(\frac{c\varepsilon - e\Delta}{c\varepsilon - (e-1)\Delta} \right) \eta(t_e) \\ &= (\kappa_c)^2 \left(\frac{c\varepsilon - e\Delta}{c\varepsilon - (e-2)\Delta} \right) \eta(t_{e-1}) \\ &= (\kappa_c)^e (1 - e\Delta(c\varepsilon)^{-1}) \eta(t_1). \end{aligned} \quad (17)$$

We are now ready to conclude this analysis with the three performance measures. From the fact that $y(t_1) = c\varepsilon$ decays to zero, the amount of samples triggered by SoD due to

the disturbance is $\alpha = \lfloor c\varepsilon\Delta^{-1} \rfloor$. Further, as $\lim_{t \rightarrow \infty} w(t) = 0$, Lemma IV.2 proves that $\eta(t)$ is GAS and thus $\eta(\infty) = 0$. The value for t_α is obtained by substituting $\mathbf{e} = \alpha$ into (16) and the last performance-measure, i.e., $\eta(t_\alpha)$, results from substituting $\eta(t_1) = \kappa_c \varepsilon$ and $\alpha = \mathbf{e} + 1$ into (17), i.e.,

$$\eta(t_\alpha) = (\kappa_c)^\alpha \varepsilon (1 - (\alpha - 1)\Delta(c\varepsilon)^{-1}).$$

B. Predictive Sampling: Impulse disturbance

For introduction, recall that the first event after the disturbance is triggered at $t_1 = \lim_{\tau \downarrow 0} t$ to result in the estimation error $\eta(t_1) = \kappa_c \varepsilon$ (see (10)). The triggering condition of PS is $|c\eta(t)| \geq \Delta$. Therefore, PS triggers “instantly” as many events until $|c\eta(t_\alpha)| < \Delta$, due to which $\tau_e \downarrow 0$ for all \mathbf{e} and $t_\alpha - t_d = \lim_{\tau \downarrow 0} \tau$. Equation (11) gives that errors are reduced with $\eta(t_{\mathbf{e}+1}) = \kappa_c a_{\tau_e} \eta(t_e) = \kappa_c a_0 \eta(t_e) = \kappa_c \eta(t_e)$ after each event and since $e(t_1) = -\kappa_c$ one obtains that $\eta(t_{\mathbf{e}+1}) = (\kappa_c)^{\mathbf{e}} \eta(t_1) = -(\kappa_c)^{\mathbf{e}+1}$. From this error equation one can thus derive the following performance measures:

$$\eta(t_\alpha) = -(\kappa_c)^\alpha \quad \text{and} \quad \alpha = \min\{n \in \mathbb{Z}_{\geq 1} \mid |c(\kappa_c)^n| < \Delta\}.$$

Similar as to SoD $\eta(t)$ is GAS and thus $\eta(\infty) = 0$.

C. Send-on-Delta: Step disturbance

Equation (12) gives that the sensor value starts to follow $y(t < t_d) = 0$ and evolves as $y(t \geq t_d) = -\frac{\varepsilon c}{a}(1 - e^{a(t-t_d)})$. Hence, after t_d the value of $y(t)$ converges from 0 to $-\frac{\varepsilon c}{a}$, implying that the amount of events triggered by SoD after t_d , i.e., the amount of Δ -levels between 0 and $|\frac{\varepsilon c}{a}|$, yields

$$\alpha = \left\lfloor \left| \varepsilon c (a\Delta)^{-1} \right| \right\rfloor.$$

The \mathbf{e} -th event is triggered when $y(t) = \mathbf{e}\Delta$. Substituting the above evolution of $y(t)$ into this equality, gives the following result for the event instants

$$-\frac{\varepsilon c}{a} \left(1 - e^{a(t_e - t_d)}\right) = \mathbf{e}\Delta \Rightarrow t_e = t_d + \frac{1}{a} \ln \left(1 + \frac{a\mathbf{e}\Delta}{\varepsilon c}\right).$$

Hence, one can derive that $t_\alpha - t_d = a^{-1} \ln \left(1 + \frac{a\alpha\Delta}{\varepsilon c}\right)$. The characterization can further be used to construct the time in between two consecutive events, i.e.,

$$\begin{aligned} \tau_e &= \frac{1}{a} \left(\ln \left(1 + \frac{a\Delta(\mathbf{e}+1)}{\varepsilon c}\right) - \ln \left(1 + \frac{a\Delta\mathbf{e}}{\varepsilon c}\right) \right) \\ &= \frac{1}{a} \ln \left(\frac{\varepsilon c + a\Delta(\mathbf{e}+1)}{\varepsilon c + a\Delta\mathbf{e}} \right). \end{aligned}$$

$$\text{and thus} \quad a_{\tau_e} = e^{a\tau_e} = \frac{\varepsilon c + a\Delta(\mathbf{e}+1)}{\varepsilon c + a\Delta\mathbf{e}}.$$

Substituting the above value for a_{τ_e} into the characterization of estimation error at events (see (14)), thus gives

$$\begin{aligned} \eta(t_{\mathbf{e}+1}) &= \kappa_c \left(a_{\tau_e} \eta(t_e) + \frac{\varepsilon}{a} (a_{\tau_e} - 1) \right) \\ &= \kappa_c \left(\frac{\varepsilon c + a\Delta(\mathbf{e}+1)}{\varepsilon c + a\Delta\mathbf{e}} \eta(t_e) + \frac{\varepsilon\Delta}{\varepsilon c + a\Delta\mathbf{e}} \right). \end{aligned}$$

In case $\gamma(n, m) := \frac{\varepsilon c + a\Delta n}{\varepsilon c + a\Delta m}$ and $\beta(m) := \frac{\varepsilon\Delta}{\varepsilon c + a\Delta m}$, where $n, m \in \mathbb{Z}$, then $\eta(t_{\mathbf{e}+1})$ can be rewritten as follows:

$$\begin{aligned} \eta(t_{\mathbf{e}+1}) &= \kappa_c \gamma(\mathbf{e}+1, \mathbf{e}) \eta(t_e) + \kappa_c \beta(\mathbf{e}), \\ &= \kappa_c^2 \gamma(\mathbf{e}+1, \mathbf{e}-1) \eta(t_{\mathbf{e}-1}) + \kappa_c^2 \gamma(\mathbf{e}+1, \mathbf{e}) \beta(\mathbf{e}-1) + \kappa_c \beta(\mathbf{e}), \\ &= \kappa_c^{\mathbf{e}} \gamma(\mathbf{e}+1, 1) \eta(t_1) + \sum_{m=1}^{\mathbf{e}} \kappa_c^{\mathbf{e}-m+1} \gamma(\mathbf{e}+1, m+1) \beta(m). \quad (18) \end{aligned}$$

At t_1 the Luenberger observer receives the first measurement after initialization $y(t_1) = \Delta$, due to which $x(t_1) = c^{-1}\Delta$. The estimation update in (3), with $x(0) = \hat{x}(0) = 0$, then gives $\hat{x}(t_1) = a_{t_1} \hat{x}(0) + \kappa (y(t_1) - c a_{t_1} \hat{x}(0)) = \kappa \Delta$ and thus $\eta(t_1) = \kappa_c \Delta c^{-1} \Delta$. Combining this result with ((18)) and $\alpha = \mathbf{e} + 1$, gives the estimation error at the last event, i.e.,

$$\eta(t_\alpha) = \kappa_c^\alpha c^{-1} \Delta \gamma(\alpha, 1) + \sum_{m=1}^{\alpha-1} \kappa_c^{\alpha-m} \gamma(\alpha, m+1) \beta(m).$$

After t_α no new event will be triggered, due to which the Luenberger observer keeps on predicting the state implying that $\eta(t)$ after t_α evolves as follows:

$$\begin{aligned} \eta(t > t_\alpha) &= a_{t-t_\alpha} \eta(t_\alpha) + \frac{\varepsilon}{a} (a_{t-t_\alpha} - 1), \\ \text{as } \lim_{t \rightarrow \infty} a_{t-t_\alpha} &= 0 \Rightarrow \eta(\infty) = \lim_{t \rightarrow \infty} \eta(t) = -\varepsilon a^{-1}. \end{aligned}$$

D. Predictive Sampling: Step disturbance

In case PS is used, triggering new events depends on the estimation error of the local observer. Based on the results presented in (14), one can conclude that the value of this local estimation error $\eta(t)$ tends to converge to $-\varepsilon a^{-1}$ after each. However, since $|-\varepsilon a^{-1}| > |\Delta c^{-1}|$, the PS-strategy keeps on triggering new events whenever $\eta(t) > |\Delta c^{-1}|$. As such, the estimation error will never converge to $-\varepsilon a^{-1}$ but will remain bounded by $|\Delta c^{-1}|$ and thus the amount of events is unbounded, i.e., $\alpha = \infty$. This further implies that there is no last event instant, i.e., $t_\alpha - t_d \rightarrow \infty$ and $\eta(t_\alpha)$ does not exist.

An interesting observation is that PS will result in a triggering mechanism with a fixed inter-sampling time. To analyze its value, first notice that $\lim_{t \uparrow t_{\mathbf{e}+1}} \eta(t) = |\Delta c^{-1}|$ and thus $\eta(t_{\mathbf{e}+1}) = \kappa_c |\Delta c^{-1}|$, for all $\mathbf{e} \in \mathbb{Z}_+$. Substituting this result into (14), gives that after the \mathbf{e}^{th} event $\eta(t)$ evolves as follows:

$$\eta(t) = a_{t-t_e} \kappa_c |\Delta c^{-1}| + \varepsilon a^{-1} (a_{t-t_e} - 1), \quad \forall t_e < t < t_{\mathbf{e}+1}.$$

Substituting the above result in the fact $\lim_{t \uparrow t_{\mathbf{e}+1}} \eta(t) = |\Delta c^{-1}|$, gives that $\lim_{t \uparrow t_{\mathbf{e}+1}} a_{t-t_e} \kappa_c |\Delta c^{-1}| + \frac{\varepsilon}{a} (a_{t-t_e} - 1) = |\Delta c^{-1}|$. This result can further be used for finding the inter-sampling time $\tau_s = t_{\mathbf{e}+1} - t_e$, since $\tau_s = \lim_{t \uparrow t_{\mathbf{e}+1}} t - t_e$, i.e.,

$$\begin{aligned} a_{\tau_s} \kappa_c |\Delta c^{-1}| + \varepsilon a^{-1} (a_{\tau_s} - 1) &= |\Delta c^{-1}|, \\ \Rightarrow a_{\tau_s} &= \frac{|\Delta c^{-1}| + \varepsilon a^{-1}}{\kappa_c |\Delta c^{-1}| + \varepsilon a^{-1}} \\ \Rightarrow \tau_s &= \frac{1}{a} \ln \left(\frac{|\Delta c^{-1}| + \varepsilon a^{-1}}{\kappa_c |\Delta c^{-1}| + \varepsilon a^{-1}} \right). \end{aligned}$$

Note that $\kappa_c |\Delta c^{-1}| < |\Delta c^{-1}|$ implies that $\frac{|\Delta c^{-1}| + \varepsilon a^{-1}}{\kappa_c |\Delta c^{-1}| + \varepsilon a^{-1}} < 1$, which together with $a < 0$ gives that $\tau_s > 0$.