Deterministic Approximation of Circular Densities with Symmetric Dirac Mixtures Based on Two Circular Moments

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***Abstract*—Circular estimation problems arise in many applications and can be addressed with the help of circular distributions. In particular, the wrapped normal and von Mises distributions are widely used in the context of circular problems. To facilitate the development of nonlinear filters, a deterministic sample-based approximation of these distributions with a so-called wrapped Dirac mixture distribution is beneficial. We propose a new closed-form solution to obtain a symmetric wrapped Dirac mixture with five components based on matching the first two circular moments. The proposed method is superior to state-of-the-art methods, which only use the first circular moment to obtain three Dirac components, because a larger number of Dirac components results in a more accurate approximation.**

Keywords*—*circular statistics, Dirac mixture, nonlinear filtering, moment matching

I. Introduction

Many estimation problems involve circular quantities, for example the orientation of a vehicle, the wind direction, or the angle of a robotic joint. Since conventional estimation algorithms perform poorly in these applications, particularly if the angular uncertainty is high, circular estimation methods such as [1], [2], [3], and [4] have been proposed. These methods use circular probability distributions that stem from the field of directional statistics [5], [6].

To facilitate the development of circular filters, sample-based approaches are commonly used, because samples (i.e., Dirac delta distributions) can easily be propagated through nonlinear functions. We distinguish deterministic and nondeterministic approaches. In the noncircular case, typical examples for deterministic approaches include the unscented Kalman filter (UKF) [7], the cubature Kalman filter [8], and the smart sampling Kalman filter (S2KF) [9]. Nondeterministic filters for the noncircular case are the particle filter [10], the Gaussian particle filter [11], and the randomized UKF [12].

We focus on deterministic approaches because they have several distinct advantages. First of all, as a result of their deterministic nature all results are reproducible. Second, the samples are placed according to a certain optimality criterion (i.e., moment matching [7] or shape approximation [13], [14]). Consequently, a much smaller number of samples is sufficient to achieve a good approximation. Third, nondeterministic approaches usually have a certain probability of causing the filtering algorithm to fail just because of a poor choice of samples. This is avoided in deterministic methods. However, deterministic methods for obtaining samples are typically more complicated and computationally demanding than nondeterministic methods.

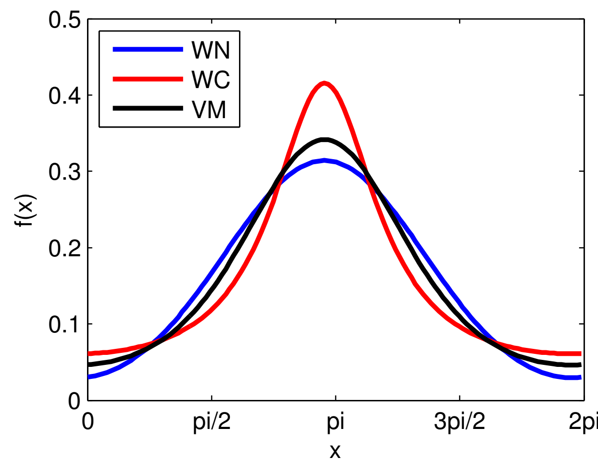


Figure 1: Probability density functions of WN, WC and VM distributions with identical first circular moment.

In our previous publication [1], we have presented a deterministic approximation for von Mises and wrapped normal distributions with three components. This approximation is based on matching the first circular moment. The first circular moment is a complex number and both a measure of location and dispersion. This approximation has already been applied to constrained object tracking [15] as well as sensor scheduling based on bearing-only measurements [16].

In this paper, we extend our previous approach [1] to match both the first and the second circular moment. This yields an approximation with five components. Even though this approximation is somewhat more complicated, it can still be computed in closed form and does not require approximations.

II. Prerequisites

In this section, we define the required probability distributions (see Fig. 1) by giving their probability density function (pdf) and introduce the concept of circular moments.

Definition 1 (Wrapped Normal Distribution)**.**
A wrapped normal (WN) distribution [5], [6] is given by the pdf

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(x - \mu + 2k\pi)^2}{2\sigma^2} \right) ,$$

where $\mu \in [0, 2\pi)$ and $\sigma > 0$ are parameters for center and uncertainty respectively.

The WN distribution is obtained by wrapping a one-dimensional Gaussian density around the unit circle. It is of particular interest because it appears as a limit distribution on the circle, i.e., in a circular setting, it is reasonable to assume that noise is WN distributed. To see this, we consider i.i.d. random variables θ_i with $\mathbb{E}(\theta_i) = 0$ and finite variance. Then the sum

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \theta_k$$

converges to a normally distributed random variable if $n \to \infty$. Consequently, the wrapped sum $(S_n \mod 2\pi)$ converges to a WN distributed random variable.

Definition 2 (Wrapped Cauchy Distribution)**.**
The wrapped Cauchy (WC) distribution [5], [6] has the pdf

$$f(x; \mu, \gamma) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{\gamma}{\gamma^2 + (x - \mu + 2k\pi)^2} ,$$

where $\mu \in [0, 2\pi)$ and $\gamma > 0$.

Similar to the WN distribution, a WC distribution is obtained by wrapping a Cauchy distribution around the circle. Unlike the WN distribution, here it is possible to simplify the infinite sum, yielding the closed-form expression

$$f(x; \mu, \gamma) = \frac{1}{2\pi} \frac{\sinh(\gamma)}{\cosh(\gamma) - \cos(x - \mu)} .$$

Definition 3 (Von Mises Distribution)**.**
A von Mises (VM) distribution [5], [6] is defined by the pdf

$$f(x; \mu, \kappa) = \frac{1}{2\pi I_0(\kappa)} \exp\left(\kappa \cos(x - \mu)\right) ,$$

where $\mu \in [0, 2\pi)$ and $\kappa > 0$ are parameters for location and concentration respectively, and $I_0(\cdot)$ is the modified Bessel function of order 0.

The modified Bessel function of integer order n is given by

$$I_n(z) = \frac{1}{\pi} \int_0^{\pi} \exp(z \cos\theta) \cos(n\theta) d\theta$$

according to [17, eq. 9.6.19]. The von Mises distribution has a similar shape as a WN distribution and is frequently used in circular statistics.

Definition 4 (Wrapped Dirac Distribution)**.**
A wrapped Dirac mixture (WD) distribution has the pdf

$$f(x; w_1, \dots, w_L, \beta_1, \dots, \beta_L) = \sum_{j=1}^{L} w_j \delta(x - \beta_j) ,$$

where L is the number of components, $\beta_1, \dots, \beta_L \in [0, 2\pi)$ are the Dirac positions, $w_1, \dots, w_L > 0$ are the weighting coefficients and δ is the Dirac delta distribution [1], [16]. We require $\sum_{j=1}^{L} w_j = 1$ to ensure that the WD distribution is normalized.

Unlike the continuous WN, WC and VM distributions, the WD distribution is a discrete distribution consisting of a certain number of Dirac delta components. These components can be seen as a set of samples and can be used to approximate a certain original density. WD densities are useful for nonlinear estimation because they can easily be propagated through nonlinear functions [1], just as Dirac mixture densities in $\mathbb{R}^n$ [9]. The WD distribution as defined above, does not contain an infinite sum for wrapping, because wrapping a Dirac distribution results in a single component according to

$$\sum_{k=-\infty}^{\infty} \delta(x + 2\pi k - \beta) = \delta((x - \beta) \mod 2\pi) .$$

We still refer to the distribution as wrapped for consistency with the WN and WC distributions.

Definition 5 (Circular Moments)**.**
The n-th circular (or trigonometric) moment of a random variable x with pdf $f(\cdot)$ is given by [5], [6]

$$m_n = \mathbb{E}(\exp(ix)^n) = \int_0^{2\pi} \exp(inx) f(x) dx ,$$

where i is the imaginary unit.

Circular moments are the circular analogon to the conventional real-valued moments $\mathbb{E}(x^n)$. Note, however, that $m_n \in \mathbb{C}$ is a complex number. For this reason, the first circular moment already describes both location and dispersion of the distribution, similar to the first two conventional real-valued moments. The argument of the complex number is analogous to the mean whereas the absolute value describes the concentration.

Lemma 1. *The circular moments of WN, WC, VM, and WD distributions are given by*

$$m_n^{WN} = \exp(in\mu - n^2\sigma^2/2) , \quad (1)$$

$$m_n^{WC} = \exp(in\mu - |n|\gamma) , \quad (2)$$

$$m_n^{VM} = \frac{I_{|n|}(\kappa)}{I_0(\kappa)} \exp(in\mu) , \quad (3)$$

$$m_n^{WD} = \sum_{j=1}^{L} w_j \exp(in\beta_j) . \quad (4)$$

Derivations can be found in [5], [6]. Here, $I_n(\cdot)$ is the modified Bessel function of order n [17]. The quotient of Bessel functions can be calculated numerically with the algorithm by [18]. Pseudo-code for this algorithm can be found in [1].

WN, WC and VM distributions are uniquely defined by their first circular moment. However, WN, WC and VM distributions with equal first moments significantly differ in their higher moments. This is illustrated in Fig. 1 and Fig. 2. This difference motivates the use of second moments in deterministic Dirac mixture approximations.

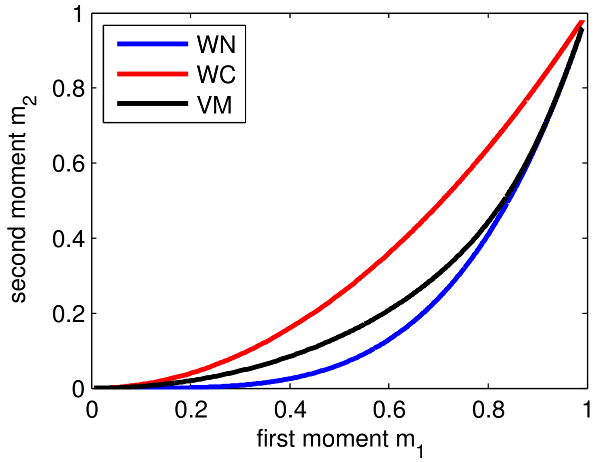


Figure 2: First circular moment of wrapped normal, wrapped Cauchy, and von Mises distributions with mean zero plotted against their second circular moment. The moments are real-valued in this case because $\mu = 0$.

III. DETERMINISTIC APPROXIMATION

In this section, we derive deterministic Dirac approximation methods for WN, WC, and VM distributions. Without loss of generality, we only consider the case $\mu = 0$ in order to simplify the calculations. In the case of $\mu \neq 0$, the samples are computed for $\mu = 0$ and subsequently shifted by μ. The moment formulas (1)-(3) simplify to $m_n^{WN} = \exp(-n^2\sigma^2/2)$, $m_n^{WC} = \exp(|n|\gamma)$, and $m_n^{VM} = \frac{I_{|n|}(\kappa)}{I_0(\kappa)}$. In particular, we find $\text{Im}(m_n^{WN}) = \text{Im}(m_n^{WC}) = \text{Im}(m_n^{VM}) = 0$, so there is no imaginary part and our calculations only involve real numbers. More general, for any circular distribution symmetric around $\mu = 0$, it holds that

$$\begin{aligned}\text{Im}\, m_n &= \int_0^{2\pi} \sin(nx) f(x) dx \\ &= \int_0^{\pi} \sin(nx) f(x) dx + \int_{\pi}^{2\pi} \sin(nx) f(x) dx \\ &= \int_0^{\pi} \sin(nx) f(x) dx + \int_0^{\pi} \sin(-nx) f(-x) dx = 0 \ .\end{aligned}$$

Keep in mind that we are still considering circular moments, not conventional moments. Furthermore, this property only holds for symmetric circular distributions and in general only the first circular moment is guaranteed to have no imaginary part.

A. First Circular Moment

First, we derive the approximation based on the first moment. We have previously presented the solution with $L = 3$ components in [1].

1) Two Components: Obviously, one WD component is not sufficient to match a given first moment, because a single component only has a single degree of freedom, whereas the first moment has two degrees of freedom. For this reason, we propose a solution with $L = 2$ components, the minimum number possible. We use symmetric WD positions $\beta_1 = -\phi, \beta_2 = \phi$, and equal weights $w_1 = w_2 = \frac{1}{2}$. For the first moment, we have

$$m_1^{WD} = \sum_{j=1}^{L} w_j \exp(i\beta_j) = \cos(\phi) \ .$$

Solving for ϕ results in $\phi = \arccos(m_1)$.

2) Three Components: Now we extend the mixture with two components by adding an additional component at the mean. Consider the WD distribution with $L = 3$ components, Dirac positions $\beta_1 = -\phi, \beta_2 = \phi, \beta_3 = 0$, and equal weights $w_1 = w_2 = w_3 = \frac{1}{3}$. For the first moment, we have

$$m_1^{WD} = \sum_{j=1}^{L} w_j \exp(i\beta_j) = \frac{1}{3}(2\cos(\phi) + 1) \ .$$

Notice that there is no imaginary part. Now, we match with the first moment m_1 of a WN or VM distribution and obtain

$$\underbrace{\frac{1}{2}(3m_1 - 1)}_{=:c_1} = \cos(\phi) \ .$$

Thus, we use $\phi = \arccos(c_1)$ to obtain a solution for the WD distribution.

B. First Two Circular Moments

Approximation based on the first two circular moments m_1 and m_2 is somewhat more involved. We consider WD distribution with $L = 5$ components and Dirac positions $\beta_1 = -\phi_1, \beta_2 = \phi_1, \beta_3 = -\phi_2, \beta_4 = \phi_2, \beta_5 = 0$ symmetric around 0. As we will show, moment matching does not allow a solution with an equally weighted Dirac mixture in general. Thus, we choose equal weights for the first four components $w_1 = w_2 = w_3 = w_4 = \frac{1-w_5}{4}$ and leave the weight w_5 of the component at zero to be determined. We will later derive constraints on the value of w_5 and see that $w_5 = \frac{1}{5}$, i.e., equal weights for all components, does not guarantee the existence of a solution in all cases.

For the first moment, we have

$$m_1^{WD} = 2\frac{1-w_5}{4}\cos(\phi_1) + 2\frac{1-w_5}{4}\cos(\phi_2) + w_5 \ ,$$

and obtain

$$\underbrace{\frac{2}{1-w_5}(m_1 - w_5)}_{=:c_1} = \cos(\phi_1) + \cos(\phi_2) \tag{5}$$

Similarly, for the second moment, we have

$$m_2^{WD} = 2\frac{1-w_5}{4}\cos(2\cdot\phi_1) + 2\frac{1-w_5}{4}\cos(2\cdot\phi_2) + w_5 \ .$$

Here, we apply the trigonometric identity $\cos(2\cdot x) = 2\cos^2(x) - 1$. After a short calculation, we obtain

$$\underbrace{\frac{1}{1-w_5}(m_2 - w_5) + 1}_{=:c_2} = \cos^2(\phi_1) + \cos^2(\phi_2) \ . \tag{6}$$

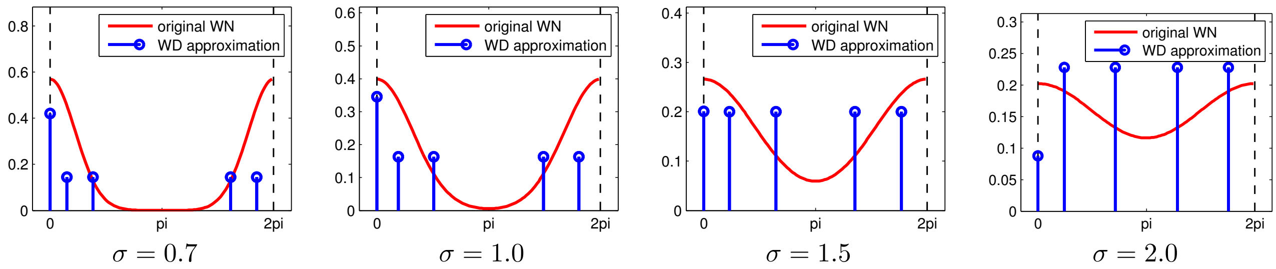


Figure 3: WD approximations for WN distributions with different values for σ. In all cases, we use $\lambda = 0.5$.

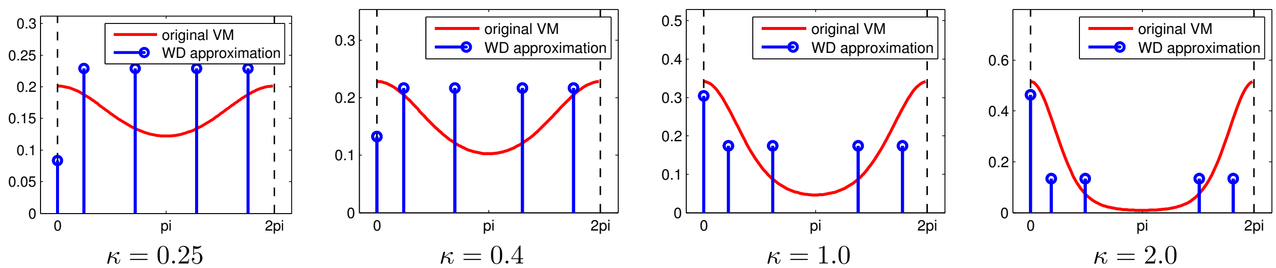


Figure 4: WD approximations for VM distributions with different values for κ. In all cases, we use $\lambda = 0.5$.

By substituting $x_1 = \cos(\phi_1), x_2 = \cos(\phi_2)$, we obtain a system of two equations

$$c_1 = x_1 + x_2 \ ,$$
$$c_2 = x_1^2 + x_2^2 \ .$$

We solve for x_1 and x_2, which yields

$$x_1 = c_1 - x_2 \ , \quad x_2 = \frac{2c_1 \pm \sqrt{4c_1^2 - 8(c_1^2 - c_2)}}{4} \ .$$

Obviously, there are two different solutions. Without loss of generality, we only consider the solution

$$x_1 = c_1 - x_2 \ , \quad x_2 = \frac{2c_1 + \sqrt{4c_1^2 - 8(c_1^2 - c_2)}}{4} \ , \tag{7}$$

because the other solution just swaps x_1 and x_2, which is equivalent for our purposes. Finally, we obtain $\phi_1 = \arccos(x_1)$ and $\phi_2 = \arccos(x_2)$.

This leaves the question of choosing the weighting coefficient w_5. The previous equations can only be evaluated if the conditions

$$-1 \leq x_i \leq 1 \ , \ i = 1, 2 \quad \text{and} \quad 4c_1^2 - 8(c_1^2 - c_2) \geq 0$$

hold. These conditions can be used to find a lower and an upper bound on w_5. These bounds are

$$w_5^{\min} = \frac{4m_1^2 - 4m_1 - m_2 + 1}{4m_1 - m_2 - 3} \ , \tag{8}$$

$$w_5^{\max} = \frac{2m_1^2 - m_2 - 1}{4m_1 - m_2 - 3} \ . \tag{9}$$

It can easily be shown that $w_5^{\min} \leq w_5^{\max}$ holds in all cases, because $0 \leq m_1 \leq 1$. Consequently, for any $0 \leq \lambda \leq 1$,

$$w_5(\lambda) = w_5^{\min} + \lambda(w_5^{\max} - w_5^{\min}) \tag{10}$$

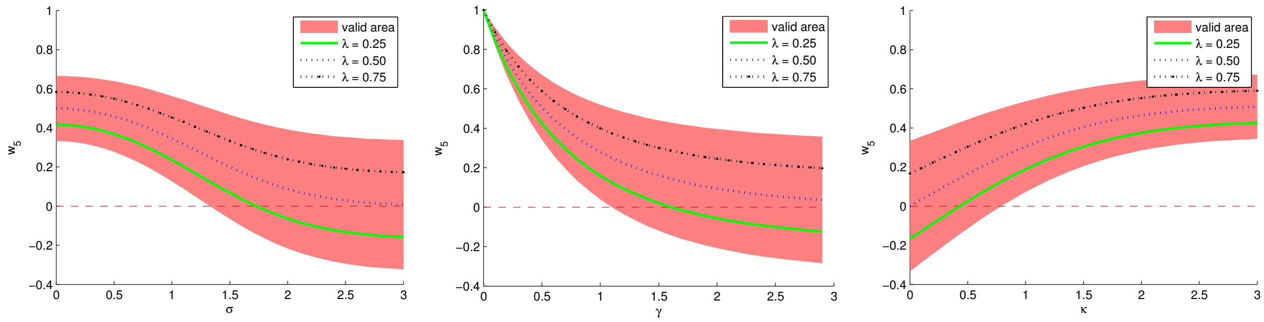
is a feasible solution. Furthermore, weights $w_5(\lambda) < 0$ are invalid because negative weights violate Kolmogorov's first axiom, i.e., the probability of any event has to be larger or equal to zero.[1] The parameter λ has to be chosen depending accordingly. The range of admissible values is illustrated in Fig. 5. It is obvious from the figure that $w_5 = \frac{1}{5}$, i.e., equal weights for all WD components, is not necessarily contained in the region of feasible values. A good choice of the parameter λ is discussed below.

IV. Properties of the Proposed Approximation

There are several noteworthy properties of the presented approximation method. Obviously, it maintains the first and second circular moment of the original density. Maintaining the first circular moment guarantees that the conversion is reversible. If we take a WN, WC, or VM distribution and approximate it with a WD distribution, we can recover the original distribution by means of moment matching. In the case of a VM distribution, we can also obtain the original distribution by maximum likelihood estimation, which coincides with the result from moment matching.

Approximating not just the first, but also the second moment has the advantage of more accurately approximating the original distribution and producing a mixture with more components. Bear in mind that the different types of distributions differ in their second moment, even if they are uniquely determined by their first moment (see Fig. 2). If we use a wrapped Dirac mixture to propagate a density through a nonlinear function, a larger number of mixture components captures the effect of the function more precisely.

[1] In practice, other filters such as the UKF [7] and the randomized UKF [12] are sometimes used with negative weights, which can give decent results, but does not have a proper probabilistic interpretation.



(a) WN distribution with parameter σ. (b) WC distribution with parameter γ. (c) VM distribution with parameter κ.

Figure 5: Feasible values for w_5 depending on a given concentration of the distribution.

One of the main advantages of the presented method is the fact that for WN and WC distributions all required operations can be evaluated in closed form. The necessary formulas (1), (2), and (5)-(10) can be evaluated in constant time and are easily implemented even on embedded hardware with limited computational capabilities. In the case of a VM distribution, the calculation of the first and second moments requires the evaluation of Bessel functions as given in (3), but all other steps (5)-(10) are still possible in closed form.

Examples for the approximation of both WN and VM densities with different concentrations are depicted in Fig. 3 and Fig. 4.

The influence of the parameter λ is illustrated in Fig. 6. If λ approaches 1, more and more weight is assigned to the Dirac component at zero whereas the other Dirac components have less influence. If, on the other hand, λ approaches zero, two of the other components move towards the center Dirac component, effectively reducing the number of Dirac components to three. As both of these effects are undesirable, λ should not be chosen too close to either zero or one.

Lemma 2 (Condition for Positive Weights)**.**
For WN and WC distributions, w_5 is positive for arbitrary concentrations if and only if $\lambda \geq 0.5$.

Proof: We calculate

$$\begin{aligned} w_5(\lambda) =& w_5^{\min} + \lambda(w_5^{\max} - w_5^{\min}) \\ =& \frac{4m_1^2 - 4m_1 - m_2 + 1}{4m_1 - m_2 - 3} \\ & + \lambda\left(\frac{2m_1^2 - m_2 - 1}{4m_1 - m_2 - 3} - \frac{4m_1^2 - 4m_1 - m_2 + 1}{4m_1 - m_2 - 3}\right) \\ =& \frac{4m_1^2 - 4m_1 - m_2 + 1}{4m_1 - m_2 - 3} + \lambda\left(\frac{-2m_1^2 + 4m_1 - 2}{4m_1 - m_2 - 3}\right) \\ =& \frac{4m_1^2 - 4m_1 - m_2 + 1 + \lambda(-2m_1^2 + 4m_1 - 2)}{4m_1 - m_2 - 3} \\ =& \frac{(4 - 2\lambda)m_1^2 + (-4 + 4\lambda)m_1 - m_2 + 1 - 2\lambda}{4m_1 - m_2 - 3} \ . \end{aligned}$$

a) WN: From (1), we obtain the relation $m_2 = m_1^4$. and substitute accordingly.

$$\begin{aligned} w_5(\lambda) =& \frac{(4 - 2\lambda)m_1^2 + (-4 + 4\lambda)m_1 - m_1^4 + 1 - 2\lambda}{4m_1 - m_1^4 - 3} \\ =& \frac{m_1^2 + 2\lambda + 2m_1 - 1}{m_1^2 + 2m_1 + 3} \end{aligned}$$

Because $m_1^2 + 2m_1 + 3 > 0$, we have

$$\begin{aligned} & w_5(\lambda) \geq 0 \\ \Leftrightarrow & m_1^2 + 2\lambda + 2m_1 - 1 \geq 0 \\ \Leftrightarrow & \lambda \geq \frac{1}{2} - 2m_1 - m_1^2 \quad \xrightarrow{m_1 \to 0} \frac{1}{2} \end{aligned}$$

and $m_1 \in (0, 1)$ shows the claim.

b) WC: From (2), we obtain the relation $m_2 = m_1^2$

$$\begin{aligned} w_5(\lambda) =& \frac{(3 - 2\lambda)m_1^2 + (-4 + 4\lambda)m_1 + 1 - 2\lambda}{4m_1 - m_1^2 - 3} \\ =& \frac{2\lambda m_1 - 2\lambda - 3m_1 + 1}{m_1 - 3} \end{aligned}$$

Because $m_1 - 3 < 0$, we have

$$\begin{aligned} & w_5(\lambda) \geq 0 \\ \Leftrightarrow & 2\lambda m_1 - 2\lambda - 3m_1 + 1 \leq 0 \\ \Leftrightarrow & \lambda(2m_1 - 2) \leq -1 + 3m_1 \\ \Leftrightarrow & \lambda \geq \frac{1}{2} \cdot \frac{1 - 3m_1}{1 - m_1} \quad \xrightarrow{m_1 \to 0} \frac{1}{2} \end{aligned}$$

and $m_1 \in (0, 1)$ shows the claim.

■

The same property holds for VM distributions as well, but the proof is more tedious because of the involved Bessel functions. For this reason, we do not give a formal proof here. There is another interesting property of the VM distribution, which we show in the following lemma.

Lemma 3 (Invariance of Likelihood)**.**
In case of a VM distribution, the likelihood function $f(\kappa|\beta_1, \dots, \beta_5, w_1, \dots w_5)$ of the sample set does not depend on the choice of λ.

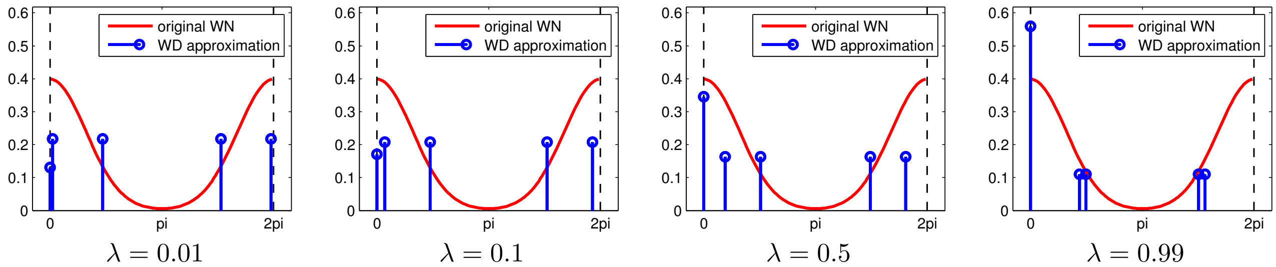


Figure 6: WD approximations for WN distributions with different values for λ. In all cases, we use $\sigma = 1$. The periodic boundary is marked by a dashed line. Note that for $\lambda \approx 0$ and $\lambda \approx 1$ the mixture degenerates to three components.

Proof: We consider the log-likelihood

$$\begin{aligned} &\log f(\kappa|\beta_1, \dots, \beta_5, w_1, \dots w_5) \\ &= \sum_{k=1}^{5} w_k \log(f(\beta_k; 0, \kappa)) \\ &= \sum_{k=1}^{5} w_k(-\log(2\pi I_0(\kappa)) + \kappa \cos(\beta_k)) \\ &= -\log(2\pi I_0(\kappa)) + \kappa \sum_{k=1}^{5} w_k \cos(\beta_k)) \ . \end{aligned}$$

As $\sum_{k=1}^{5} w_k \cos(\beta_k))$ is the first moment m_1, we obtain

$$\log f(\kappa|\beta_1, \dots, \beta_5, w_1, \dots w_5) = -\log(2\pi I_0(\kappa)) + \kappa m_1 \ ,$$

which is independent of the weights and, consequently, of λ. ∎

Based on these results, we suggest to use $\lambda = 0.5$, as we do in all of our examples. The value $\lambda = 0.5$ is a reasonable choice for both high and low concentrations and it guarantees $w_5 > 0$ in all cases.

Even though we only presented an approximation for WN and VM distributions so far, the presented approach can easily be generalized to any circular probability distribution whose first and second circular moment can be calculated.

V. Evaluation

We evaluate the proposed deterministic approximation methods by determining the error when propagating through a nonlinear function. The weighting parameter is chosen as $\lambda = 0.5$ in all simulations. For our evaluation, we consider the function $g : S^1 \to S^1$ with

$$g(x) = x + c\sin(x) \mod 2\pi$$

for some constant $0 < c < 1$. This function is continuous because $g(0) = \lim_{x \to 2\pi} g(x)$. We have $g'(x) = 1 + c\cos(x)$, which is positive any for $|c| < 1$, i.e., the function g is strictly increasing and thus injective[2]. Varying the value of c allows us to control how strong the nonlinearity is.

Now, we assume a random variable A is distributed according to a WN probability distribution $f(x; \mu, \sigma)$. We propagate A through the nonlinear function g and seek to obtain a WN distribution $f(x; \mu_g, \sigma_g)$ that approximates the distribution of $g(A)$. The exact distribution is given by

$$f_g(x) = \frac{f(g^{-1}(x); \mu, \sigma)}{g'(x)} \ .$$

This distribution is not a WN distribution, but can be approximated by one. Furthermore, it can not be evaluated in closed form because $g^{-1}(\cdot)$ can only be calculated numerically.

To approximate the true distribution with a WN distribution, we proceed as follows. First, we deterministically approximate the prior distribution with a WD mixture as presented in this paper. Then we propagate all of the samples through the nonlinear function g and finally fit a WN distribution to the resulting WD mixture. This process is illustrated in Fig. 7.

We calculate the optimal WN approximation f^{WN} of the posterior density f_g by matching the first circular moment of f_g. Then, we use the Kullback-Leibler divergence

$$\int_0^{2\pi} f^{opt}(x) \log\left(\frac{f^{opt}(x)}{f(x; \mu_g, \sigma_g)}\right) dx$$

between the f^{opt} and the fitted WN to quantify the information loss by this approximation. The results for $\mu = 0$ and different values of nonlinearity c as well as uncertainty σ are depicted in Fig. 8. It can be seen that the approximation quality is consistently higher with five rather than three or even two WD components by a significant amount. This large increase in accuracy easily justifies the very reasonable increase in computational effort.

VI. Conclusion

We have presented a new method to deterministically approximate circular distributions by a wrapped Dirac mixture based on matching the first and second circular moment. The proposed approach is applicable to a variety of circular distributions, in particular the widely used wrapped normal and von Mises distributions. Because all expressions can be evaluated in closed form, the algorithm requires little computational power and is suitable for implementation even on embedded devices.

One might wonder if the presented algorithm can easily be generalized to higher moments, but such a generalization is nontrivial. This is due to the fact that preserving n moments involves finding the roots of polynomials of degree n. Analytical

[2]The proposed approach is not limited to injective or continous functions. However, we use such a function because these properties simplify the calculation of the true posterior density.

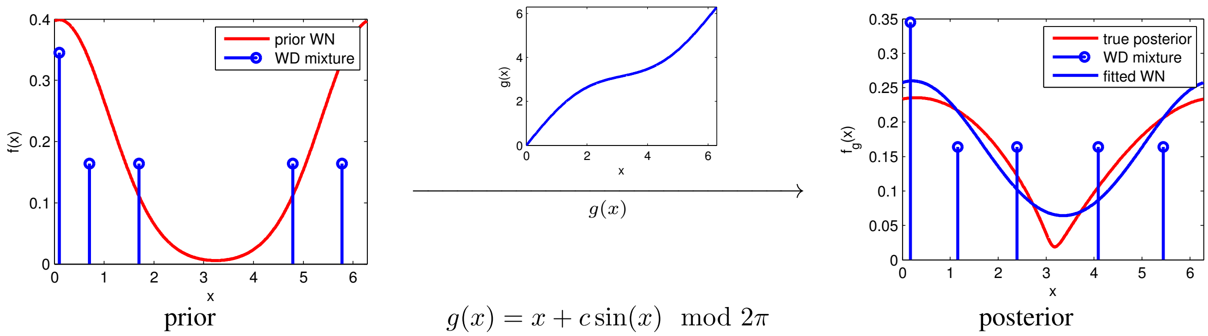
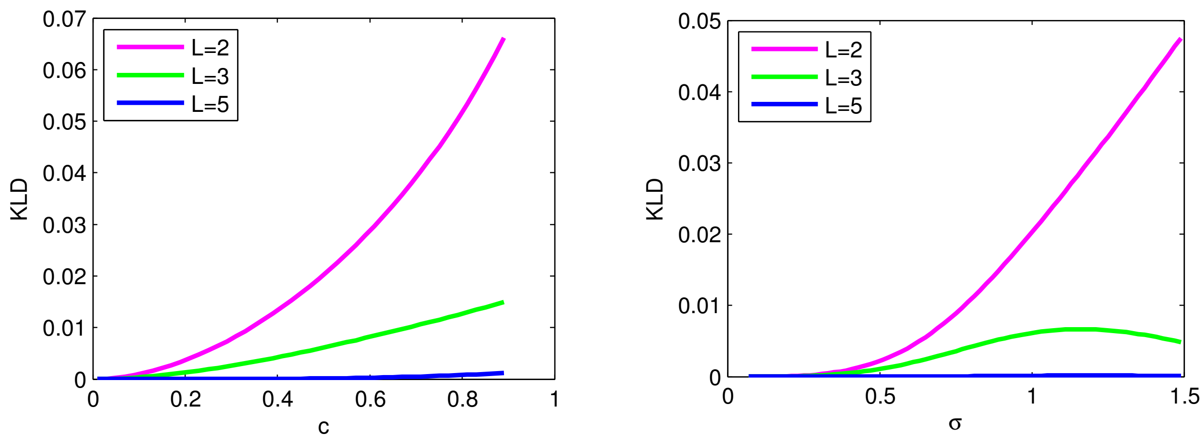


Figure 7: Propagation of a prior WN distribution with parameters $\mu = 0.1, \sigma = 1$ through a nonlinear function g by means of the proposed deterministic WD mixture approximation with five components. In this example, we use $c = 0.7$.



(a) fixed $\sigma = 1$

(b) fixed $c = 0.5$

Figure 8: Kullback-Leibler divergence between best WN that approximates the posterior and the WD-based WN approximation. We compare $L = 2$, $L = 3$ as well as $L = 5$ WD components for different degrees of nonlinearity c and uncertainty σ.

solutions only exist for polynomials of order ≤ 4 and are very complicated for $n = 3$ and $n = 4$.

Future work may include the approximation of circular probability distributions based on their shape rather than circular moments. This allows the use of more samples, and thus, a more accurate propagation. Furthermore, we plan to use the presented wrapped Dirac approximation in nonlinear circular filters.

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