

Covariance Intersection in State Estimation of Dynamical Systems

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Abstract—The Covariance Intersection algorithm linearly combines estimates when the cross-correlations between their errors are unknown. It provides a fused estimate and an upper bound of the corresponding mean square error matrix. The weights of the linear combination are designed in order to minimise the upper bound. This paper analyses the optimal weights in relation to state estimation of dynamical systems. It is shown that the use of the optimal upper bound in a standard recursive filtering does not lead to optimal upper bounds in subsequent processing steps. Unlike the fusion under full knowledge, the fusion under unknown cross-correlations can fuse the same information differently, depending on the independent information that will be available in the future.

Keywords—information fusion, decentralised estimation, unknown cross-correlations, Covariance Intersection, dynamical systems

I. INTRODUCTION

If many sensors or sensor groups employ their own local measurements to estimate the state of a common system [1], [2], the information fusion problem arises. In distributed fusion [3]–[8], the estimation problem is considered in a top-down perspective, i.e. with one central filter and subordinate local ones. The fusion performed in the central filter maximises the quality of the fused estimator, while the problem structure is usually fixed. The bottom-up perspective prevails in the decentralised fusion. The local filters are self-sufficient and the fusion is optional. This approach is better suited for complex problems, where it is impractical to compute and store the cross-correlations between errors of local estimates. However, the missing knowledge prevents the derivation of the fused estimator quality, likewise its maximisation.

Covariance Intersection [9]–[13] considers all admissible cross-correlations and the fused estimator achieves the best guaranteed quality. The actual quality of the fused estimator is not decisive, the focal point is that it stays better than the fusion-reported quality. The quality of estimators is measured by mean square error matrices. The consideration of all (also unknown) cross-correlations is substantially effectuated by the proposition of matrices that are greater than the unknown joint mean square error matrix, irrespective of the actual value of the cross-correlation. The upper bounds are ordinarily proposed as block-diagonal matrices, nevertheless, more elaborated solutions are available [14]–[19]. The processing [20] of the proposed upper bounds relies on properties of matrix operations. The optimal fused estimator is the one that

guarantees the minimal upper bound of its mean square error matrix. However, since no ordering that is meaningful in the mean square error sense can be defined on arbitrary matrices, a scalar criterion has to be optimised.

The criterion is usually chosen to be the determinant or the trace of the upper bound. It is also possible to minimise the trace of the product of the upper bound and a positive semidefinite weighting matrix. Unlike in linear estimators that operate under known cross-correlations, optimising the fusion gains according to specific dimensions of the estimated quantity yields different results than optimising the full upper bound.

The Kalman filter minimises the mean square error matrix of linear estimators. Starting with an initial minimum mean square error estimate of the state, it combines estimates with noisy measurements at the time the measurements are observed. A virtue of the Kalman filter is that the optimal estimator can be expressed recursively, i.e. there is no need to store past measurements.

Replacing the initial optimal estimate by a Covariance Intersection based estimate and the exact mean square error matrix by an upper bound, state estimators in subsequent time can be proposed by applying the Kalman filter formulas on the estimate and the upper bound. In this case, the obtained error matrices are upper bounds of the underlying mean square error matrices. However, the optimality of such estimators from the upper bound perspective has not been considered in literature, it has not been discussed whether the optimal estimators require to store the past measurements or not.

Therefore, the goal of this paper is to prospect the area. The goal is to compare the recursive approach that combines Covariance Intersection and the Kalman filter with a batch approach to the fusion of estimates and subsequent measurements.

The problem is formulated in Section II. Section III performs an analysis of the thought to compute the optimal solution recursively. Section IV provides numerical examples, Section V concludes the paper.

II. PROBLEM SETTINGS

Let $\mathcal{X}_0, \mathcal{W}_k, \mathcal{V}_k, k = 0, 1, \dots$ denote independent random vectors. The matrix $\mathbb{P}_0 = \mathbb{E}\{(\mathcal{X}_0 - \hat{x}_0)(\mathcal{X}_0 - \hat{x}_0)^T\}$ is defined, where \hat{x}_0 is a known constant vector of dimension $\dim(\mathcal{X}_0)$.

Suppose that $\mathbf{Q} = \mathbb{E}\{\mathcal{W}_k \mathcal{W}_k^T\}$ are known constant matrices. Let the random vectors depict a linear stochastic system,

$$\mathcal{X}_{k+1} = \mathbf{F}\mathcal{X}_k + \mathbf{G}\mathcal{W}_k, \quad (1)$$

$$\mathcal{Z}_k = \mathbf{H}\mathcal{X}_k + \mathcal{V}_k, \quad (2)$$

where the matrices \mathbf{F} , \mathbf{G} and \mathbf{H} are known and of appropriate dimensions. Consider the following partition

$$\mathcal{Z}_k = \begin{bmatrix} \mathcal{Z}_k^A \\ \mathcal{Z}_k^B \end{bmatrix}, \mathbf{H} = \begin{bmatrix} \mathbf{H}^A \\ \mathbf{H}^B \end{bmatrix}, \mathcal{V}_k = \begin{bmatrix} \mathcal{V}_k^A \\ \mathcal{V}_k^B \end{bmatrix}, \quad (3)$$

with known measurement noise matrices $\mathbf{R}^A = \mathbb{E}\{\mathcal{V}_k^A (\mathcal{V}_k^A)^T\}$ and $\mathbf{R}^B = \mathbb{E}\{\mathcal{V}_k^B (\mathcal{V}_k^B)^T\}$.

Let $\Omega_{\mathcal{X}}$ denote the domain of a random vector \mathcal{X} . Consider functions,

$$\hat{x}_k^A : \Omega_{\mathcal{Z}_0^A} \times \dots \times \Omega_{\mathcal{Z}_k^A} \rightarrow \Omega_{\mathcal{X}_k}, \quad (4)$$

with corresponding random vectors $\hat{\mathcal{X}}_k^A = \hat{x}_k^A(\mathcal{Z}_0^A, \dots, \mathcal{Z}_k^A)$. Then the matrices $\mathbb{P}_k^A = \mathbb{E}\{(\mathcal{X}_k - \hat{\mathcal{X}}_k^A)(\mathcal{X}_k - \hat{\mathcal{X}}_k^A)^T\}$ are independent of realisations of \mathcal{X}_k and $\hat{\mathcal{X}}_k^A$. The functions \hat{x}_k^A are called estimators, the random vectors $\hat{\mathcal{X}}_k^A$ will be referred as estimates. The mean square error matrix \mathbb{P}_k^A is given by the expectation of a matrix term taken over \mathcal{X}_k as well as over $\mathcal{Z}_0^A, \dots, \mathcal{Z}_k^A$ inherited in $\hat{\mathcal{X}}_k^A$. The functions \hat{x}_k^B , random vectors $\hat{\mathcal{X}}_k^B$ and matrices \mathbb{P}_k^B are defined analogously.

The problem of fusion is to find a function

$$\hat{x}_k^F : \Omega_{\mathcal{X}_k} \times \Omega_{\mathcal{X}_k} \rightarrow \Omega_{\mathcal{X}_k}, \quad (5)$$

such that the random vector $\hat{\mathcal{X}}_k^F = \hat{x}_k^F(\hat{\mathcal{X}}_k^A, \hat{\mathcal{X}}_k^B)$ is close to \mathcal{X}_k in a predetermined sense. In this paper, only linear estimators are considered. It is supposed that the cross-correlation matrix $\mathbb{P}_k^{AB} = \mathbb{E}\{(\mathcal{X}_k - \hat{\mathcal{X}}_k^A)(\mathcal{X}_k - \hat{\mathcal{X}}_k^B)^T\}$ is unknown. Let

$$\mathbb{\Pi}_k^F - \mathbb{E}\{(\mathcal{X}_k - \hat{\mathcal{X}}_k^F)(\mathcal{X}_k - \hat{\mathcal{X}}_k^F)^T\} \geq 0 \quad (6)$$

be the constraint that defines upper bounds of mean square error matrices, where the inequality ≥ 0 is to be understood in a positive semidefinite sense. Then, the optimisation criterion is some measure on matrices such as the determinant or trace. Hence, \hat{x}_k^F is optimal when the bound $\mathbb{\Pi}_k^F$ is minimised.

The problem posed in this paper is to relate the fusion operation and the dynamics of the considered system. More precisely, upper bounds of mean square error matrices of estimators are examined not only in the fusion, but also in combination with subsequent filter steps. Let κ denote a time horizon, $\kappa = 1, 2, \dots$. Then, the problem can be posed as optimising the linear estimators

$$\hat{x}_{k,k;\kappa}^{A,B;A} : \Omega_{\mathcal{X}_k} \times \Omega_{\mathcal{X}_k} \times \Omega_{\mathcal{Z}_{k+1}^A} \times \dots \times \Omega_{\mathcal{Z}_{k+\kappa}^A} \rightarrow \Omega_{\mathcal{X}_{k+\kappa}}, \quad (7)$$

$\hat{\mathcal{X}}_{k,k;\kappa}^{A,B;A} = \hat{x}_{k,k;\kappa}^{A,B;A}(\hat{\mathcal{X}}_k^A, \hat{\mathcal{X}}_k^B, \mathcal{Z}_{k+1}^A, \dots, \mathcal{Z}_{k+\kappa}^A)$ for all κ , in the upper bound sense that has been discussed above. In particular, it is investigated whether the linear estimator $\hat{x}_{k,k;\kappa}^{A,B;A}$ that minimises $\mathbb{\Pi}_{k,k;\kappa}^{A,B;A}$ can be computed recursively for all κ in a single run. The recursive estimator is given by

$$\hat{x}_{k;\kappa}^{F;A} : \Omega_{\mathcal{X}_{k+\kappa-1}} \times \Omega_{\mathcal{Z}_{k+\kappa}^A} \rightarrow \Omega_{\mathcal{X}_{k+\kappa}}, \quad (8)$$

$\hat{\mathcal{X}}_{k;\kappa}^{F;A} = \hat{x}_{k;\kappa}^{F;A}(\hat{\mathcal{X}}_{k;\kappa-1}^{F;A}, \mathcal{Z}_{k+\kappa}^A)$, and initialised by $\hat{\mathcal{X}}_{k;0}^{F;A} = \hat{\mathcal{X}}_k^F$, while only the corresponding upper bound $\mathbb{\Pi}_k^F$ is available. The

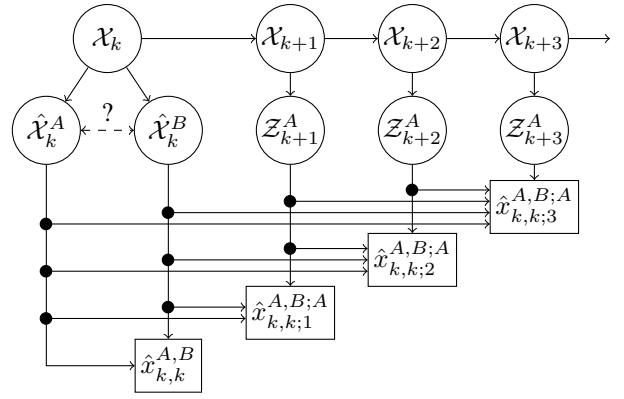


Fig. 1. Batch approach for the fusion of estimates $\hat{\mathcal{X}}_k^A, \hat{\mathcal{X}}_k^B$ and the measurements $\mathcal{Z}_{k+1}^A, \dots, \mathcal{Z}_{k+\kappa}^A$.

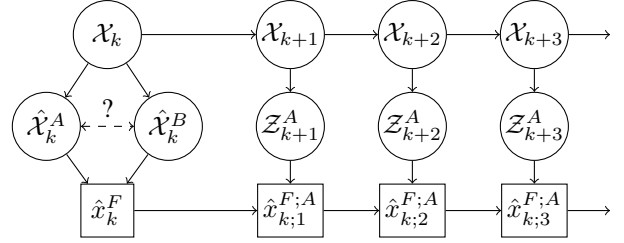


Fig. 2. Recursive approach for the fusion of estimates $\hat{\mathcal{X}}_k^A, \hat{\mathcal{X}}_k^B$ and the measurements $\mathcal{Z}_{k+1}^A, \dots, \mathcal{Z}_{k+\kappa}^A$.

batch and recursive approaches are illustrated in Fig. 1 and 2. The circles denote random vectors, rectangles denote functions, the dashed arrows indicate unknown cross-correlations.

The question is whether the (upper bound) optimal batch estimator $\hat{x}_{k,k;\kappa}^{A,B;A}$ can be obtained by concatenating the Covariance Intersection estimator \hat{x}_k^F and the Kalman filter $\hat{x}_{k;\kappa}^{F;A}$. On a more general level, the question is whether fusion methods that bound unknown correlations are invariant under an increasing amount of information.

III. COMPARISON OF THE RECURSIVE ESTIMATOR WITH THE BATCH ESTIMATOR

In this section, it is demonstrated that the optimal result in the presence of unknown correlations cannot be obtained by a recursive estimator. Although the estimators are linear, the missing knowledge of cross-correlations leads to a non-recursive solution. The optimal upper bound of the actual mean square error matrix does not necessarily lead to an optimal upper bound after further prediction and filtering steps. This is shown in the following by considering the simple case with $\kappa = 1$ and empty \mathcal{Z}_{k+1}^A .

Recursive approach: First, the linear fusion of the local estimates $\hat{\mathcal{X}}_k^A$ and $\hat{\mathcal{X}}_k^B$ is considered. The fused estimator $\hat{\mathcal{X}}_k^F$ is given by

$$\hat{x}_k^F : \hat{\mathcal{X}}_k^F = \mathbf{W}_k^A \hat{\mathcal{X}}_k^A + \mathbf{W}_k^B \hat{\mathcal{X}}_k^B, \quad \mathbf{W}_k^A + \mathbf{W}_k^B = \mathbf{I}, \quad (9)$$

where the equality condition on the matrix weights \mathbf{W}_k^A and \mathbf{W}_k^B is required in order to be able to express an upper bound

\mathbb{P}_k^F of the mean square error matrix \mathbb{P}_k^F , see (6), in terms of the available matrices. The matrix \mathbb{P}_k^F is given by

$$\mathbb{P}_k^F = \mathbf{W}_k^{\{A,B\}} \mathbb{P}_k^{\{A,B\}} (\mathbf{W}_k^{\{A,B\}})^T, \quad (10)$$

where the joint weight $\mathbf{W}_k^{\{A,B\}}$ and the joint mean square error matrix $\mathbb{P}_k^{\{A,B\}}$ are given by

$$\mathbf{W}_k^{\{A,B\}} = [\mathbf{W}_k^A \quad \mathbf{W}_k^B], \quad \mathbb{P}_k^{\{A,B\}} = \begin{bmatrix} \mathbb{P}_k^A & \mathbb{P}_k^{AB} \\ (\mathbb{P}_k^{AB})^T & \mathbb{P}_k^B \end{bmatrix}. \quad (11)$$

The upper bound \mathbb{P}_k^F can be obtained by replacing the unknown matrix $\mathbb{P}_k^{\{A,B\}}$ by its upper bound $\mathbb{P}_k^{\{A,B\}}$. This results in

$$\mathbb{P}_k^F = \mathbf{W}_k^{\{A,B\}} \mathbb{P}_k^{\{A,B\}} (\mathbf{W}_k^{\{A,B\}})^T. \quad (12)$$

The proposed upper bounds $\mathbb{P}_k^{\{A,B\}}$ are given by [15], [16], [17] as

$$\mathbb{P}_k^{\{A,B\}}(\omega) = \begin{bmatrix} \frac{1}{\omega} \mathbb{P}_k^A & \mathbf{0} \\ \mathbf{0} & \frac{1}{1-\omega} \mathbb{P}_k^B \end{bmatrix}, \quad (13)$$

where ω , $0 \leq \omega \leq 1$, is a free parameter that is to be determined with respect to the considered optimality criterion. Under unknown correlations, it is not the mean square error that is optimised, but the criterion is based on the upper bound \mathbb{P}_k^F and is a function of ω and \mathbf{W}_k^A (the matrix weight \mathbf{W}_k^B is given by (9)).

By completing the square, the upper bound \mathbb{P}_k^F is given by

$$\begin{aligned} \mathbb{P}_k^F &= \mathbf{\Delta}_k^F(\omega) \left(\frac{1}{\omega} \mathbb{P}_k^A + \frac{1}{1-\omega} \mathbb{P}_k^B \right) (\mathbf{\Delta}_k^F(\omega))^T + \\ &+ \frac{1}{1-\omega} \mathbb{P}_k^B - \frac{1}{1-\omega} \mathbb{P}_k^B \left(\frac{1}{\omega} \mathbb{P}_k^A + \frac{1}{1-\omega} \mathbb{P}_k^B \right)^{-1} \frac{1}{1-\omega} \mathbb{P}_k^B = \\ &= \mathbf{\Delta}_k^F(\omega) \left(\frac{1}{\omega} \mathbb{P}_k^A + \frac{1}{1-\omega} \mathbb{P}_k^B \right) (\mathbf{\Delta}_k^F(\omega))^T + \\ &+ (\omega (\mathbb{P}_k^A)^{-1} + (1-\omega) (\mathbb{P}_k^B)^{-1})^{-1}, \end{aligned} \quad (14)$$

$$\begin{aligned} \mathbf{\Delta}_k^F(\omega) &= \mathbf{W}_k^A - \frac{1}{1-\omega} \mathbb{P}_k^B \left(\frac{1}{\omega} \mathbb{P}_k^A + \frac{1}{1-\omega} \mathbb{P}_k^B \right)^{-1} = \\ &= \mathbf{W}_k^A - (\omega (\mathbb{P}_k^A)^{-1} + (1-\omega) (\mathbb{P}_k^B)^{-1})^{-1} \omega (\mathbb{P}_k^A)^{-1}, \end{aligned} \quad (15)$$

where the second equalities in (14) and (15) result from the matrix inversion lemma. The optimal weight \mathbf{W}_k^A is given by setting $\mathbf{\Delta}_k^F(\omega)$ to be zero, where the parameter ω is the one that minimises the constant term in the completion to square (14) over the domain of ω , see also Covariance Intersection in [9]–[20].

Having obtained the fused estimate $\hat{\mathcal{X}}_k^F$ with the optimal upper bound \mathbb{P}_k^F of the related mean square error matrix, a predicted estimate $\hat{\mathcal{X}}_{k;1}^{F;-}$ is subsequently computed. It is supposed that, in retrospect, the fused estimate $\hat{\mathcal{X}}_k^F$ cannot be improved any more. Then, the predictive estimator $\hat{\mathcal{X}}_{k;1}^{F;-}$ and the corresponding upper bound $\mathbb{P}_{k;1}^{F;-}$ are given by

$$\hat{\mathcal{X}}_{k;1}^{F;-} : \hat{\mathcal{X}}_{k;1}^{F;-} = \mathbf{F} \hat{\mathcal{X}}_k^F, \quad \mathbb{P}_{k;1}^{F;-} = \mathbf{F} \mathbb{P}_k^F \mathbf{F}^T + \mathbf{G} \mathbf{Q} \mathbf{G}^T. \quad (16)$$

In summary, the recursive approach to obtain an estimate of \mathcal{X}_{k+1} is to fuse the local estimates $\hat{\mathcal{X}}_k^A$ and $\hat{\mathcal{X}}_k^B$ by (9) at first, while \mathbf{W}_k^A is given such that the quadratic term in (14) is zero and is a function of the parameter ω , that is chosen to minimise the constant term in (14) in a predetermined sense. Subsequently, the approach propagates the fused estimate $\hat{\mathcal{X}}_k^F$ in time in the standard way (16).

Batch approach: In the considered simple case $\kappa = 1$ with empty \mathcal{Z}_{k+1}^A , the analysis is facilitated. Assuming an invertible matrix \mathbf{F} , the predictive estimator of \mathcal{X}_{k+1} is obtained by inverting the order of time propagation and fusion without loss of generality. Using the available knowledge, the analysis assumes that the estimator $\hat{\mathcal{X}}_{k,k;1}^{A,B;-}$ first propagates the local estimates and the blocks of the joint upper bound $\mathbb{P}_k^{\{A,B\}}$ (13) analogically to (16) as

$$\hat{\mathcal{X}}_{k;1}^{A;-} = \mathbf{F} \hat{\mathcal{X}}_k^A, \quad \hat{\mathcal{X}}_{k;1}^{B;-} = \mathbf{F} \hat{\mathcal{X}}_k^B, \quad (17)$$

$$\begin{aligned} \mathbb{P}_{k;1}^{\{A,B\};-}(\omega) &= \\ &= \begin{bmatrix} \frac{1}{\omega} \mathbf{F} \mathbb{P}_k^A \mathbf{F}^T + \mathbf{G} \mathbf{Q} \mathbf{G}^T & \mathbf{G} \mathbf{Q} \mathbf{G}^T \\ \mathbf{G} \mathbf{Q} \mathbf{G}^T & \frac{1}{1-\omega} \mathbf{F} \mathbb{P}_k^B \mathbf{F}^T + \mathbf{G} \mathbf{Q} \mathbf{G}^T \end{bmatrix}. \end{aligned} \quad (18)$$

The instrumental random vectors (17) are subsequently linearly fused. So, the fused estimator $\hat{\mathcal{X}}_{k,k;1}^{A,B;-}$ is given by

$$\begin{aligned} \hat{\mathcal{X}}_{k,k;1}^{A,B;-} : \hat{\mathcal{X}}_{k,k;1}^{A,B;-} &= \mathbf{W}_{k;1}^{A;-} \hat{\mathcal{X}}_{k;1}^{A;-} + \mathbf{W}_{k;1}^{B;-} \hat{\mathcal{X}}_{k;1}^{B;-}, \\ \mathbf{W}_{k;1}^{A;-} + \mathbf{W}_{k;1}^{B;-} &= \mathbf{I}. \end{aligned} \quad (19)$$

The upper bound $\mathbb{P}_{k,k;1}^{A,B;-}$ of its mean square error matrix is obtained by the same steps that have been used in the derivation of \mathbb{P}_k^F . Completing the square leads to

$$\begin{aligned} \mathbb{P}_{k,k;1}^{A,B;-} &= \mathbf{\Delta}_{k,k;1}^{A,B;-}(\omega) \mathbf{F} \left(\frac{\mathbb{P}_k^A}{\omega} + \frac{\mathbb{P}_k^B}{1-\omega} \right) \mathbf{F}^T (\mathbf{\Delta}_{k,k;1}^{A,B;-}(\omega))^T + \\ &+ \mathbf{F} \frac{\mathbb{P}_k^B}{1-\omega} \mathbf{F}^T + \mathbf{G} \mathbf{Q} \mathbf{G}^T - \\ &- \mathbf{F} \frac{\mathbb{P}_k^B}{1-\omega} \mathbf{F}^T \left(\mathbf{F} \left(\frac{\mathbb{P}_k^A}{\omega} + \frac{\mathbb{P}_k^B}{1-\omega} \right) \mathbf{F}^T \right)^{-1} \mathbf{F} \frac{\mathbb{P}_k^B}{1-\omega} \mathbf{F}^T, \end{aligned} \quad (20)$$

$$\begin{aligned} \mathbf{\Delta}_{k,k;1}^{A,B;-}(\omega) &= \\ &= \mathbf{W}_{k;1}^{A;-} - \frac{1}{1-\omega} \mathbf{F} \mathbb{P}_k^B \mathbf{F}^T \left(\mathbf{F} \left(\frac{\mathbb{P}_k^A}{\omega} + \frac{\mathbb{P}_k^B}{1-\omega} \right) \mathbf{F}^T \right)^{-1}. \end{aligned} \quad (21)$$

The optimal weighting matrix $\mathbf{W}_{k;1}^{A;-}$ is given by setting $\mathbf{\Delta}_{k,k;1}^{A,B;-}(\omega)$ to be zero, where the parameter ω is the one that minimises the constant term in (20) over the domain of ω , see also Split Covariance Intersection in [14], [16].

Comparison: The batch estimator $\hat{\mathcal{X}}_{k,k;1}^{A,B;-}$ as well as the recursive estimator $\hat{\mathcal{X}}_{k;1}^{F;-}$ linearly combine the local estimates $\hat{\mathcal{X}}_k^A$ and $\hat{\mathcal{X}}_k^B$. The matrix weights at $\hat{\mathcal{X}}_k^A$ are given by (17) and (19) as $\mathbf{W}_{k;1}^{A;-} \mathbf{F}$ and by (9) and (16) as $\mathbf{F} \mathbf{W}_k^A$, while the matrix weights at $\hat{\mathcal{X}}_k^B$ are given by $\mathbf{W}_{k;1}^{B;-} = (\mathbf{I} - \mathbf{W}_{k;1}^{A;-}) \mathbf{F}$ and $\mathbf{W}_k^B = \mathbf{F} (\mathbf{I} - \mathbf{W}_k^A)$ for the batch and recursive approaches, respectively.

If the parameter ω is fixed at time k and the weight terms \mathbf{W} are obtained by setting the $\mathbf{\Delta}$ terms in (21) and (15) to zero, then one can also compare (20) with (14) and (16). For invertible matrices \mathbf{F} and the fixed parameter ω , one easily finds out that the weights at $\hat{\mathcal{X}}_k^A$ are the same, as well as the upper bounds $\mathbb{P}_{k,k;1}^{A,B;-}(\omega)$ and $\mathbb{P}_{k;1}^{F;-}(\omega)$, for the batch and recursive approaches. Thus, the construction of the instrumental estimates (17) is actually cancelled.

The recursive approach does require to fix ω at time k , since the recursion cannot start with undetermined $\hat{\mathcal{X}}_k^F$, see Fig. 2. On the other side, the batch approach does not require

the hard decision on the value of ω at time k , see Fig. 1. Instead, the optimal value of ω is searched afresh at each time $k + \kappa$, and the criterion is different from that of time k . Thus, the batch estimator differs from the recursive one in general. Opposed to the estimation with full information, the recursive approach does not maintain the optimality of the estimators in the sense of the minimal upper bound of the mean square error matrix. The best representation of ignorance of a cross-correlation matrix at time k is dependent on the time $k + \kappa$ when the upper bounds are of the interest. Numerical examples are given in the following section.

IV. EXAMPLES

To demonstrate the influence of the information obtained in the future on the best representation of ignorance, a simple static system is considered first. Subsequently, a dynamical system will be used to discuss some asymptotic properties.

A. Simple static system

Let the linear stochastic system (1)–(3) be a static one, let the dimension of \mathcal{X}_0 be 2 and consider no prior information,

$$\mathbf{F} = \mathbf{I}, \quad \mathbf{G} = [0, 0]^T, \quad \mathbf{Q} = 0, \quad \mathbb{P}_0 \rightarrow \infty \cdot \mathbf{I}, \quad (22)$$

while \mathbf{I} denotes the identity matrix of order 2. Let both the components of the state be measured by both filters A and B , while the filter A measures the first component with a lower error than the filter B and the second component with a greater error than the filter B ,

$$\mathbf{H}^A = \mathbf{I}, \quad \mathbf{R}^A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{H}^B = \mathbf{I}, \quad \mathbf{R}^B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}. \quad (23)$$

Further, let the local estimators \hat{x}_0^A, \hat{x}_0^B be given by the local measurements directly, $\hat{x}_0^A : \hat{\mathcal{X}}_0^A = \mathcal{Z}_0^A, \hat{x}_0^B : \hat{\mathcal{X}}_0^B = \mathcal{Z}_0^B$. The local upper bounds $\mathbb{P}_0^A, \mathbb{P}_0^B$ are given directly by the mean square error matrices $\mathbb{P}_0^A, \mathbb{P}_0^B$ that are equal to \mathbf{R}^A and \mathbf{R}^B respectively, $\mathbb{P}_0^A = \mathbb{P}_0^A = \mathbf{R}^A, \mathbb{P}_0^B = \mathbb{P}_0^B = \mathbf{R}^B$.

Considering a fixed value of ω in (13), the corresponding optimal weight \mathbf{W}_0^A and the corresponding upper bound \mathbb{P}_0^F are given by

$$\mathbf{W}_0^A(\omega) = \begin{bmatrix} \frac{2\omega}{1+\omega} & 0 \\ 0 & \frac{\omega}{2-\omega} \end{bmatrix}, \quad \mathbb{P}_0^F(\omega) = \begin{bmatrix} \frac{2}{1+\omega} & 0 \\ 0 & \frac{2}{2-\omega} \end{bmatrix}. \quad (24)$$

It is easy to see that both the determinant and the trace of $\mathbb{P}_0^F(\omega)$ are minimised by $\omega = \frac{1}{2}$. This result have been expected, since the settings (23) as well as the criterion show a symmetry. So, the parameter ω is chosen in such a way that the components \mathcal{X}_0 are estimated with equal bounds of uncertainty.

However, the fusion will be different, if it is known that another measurement given by \mathbf{H}^A and \mathbf{R}^A will be available in the future. This holds even under the assumption that the measurement error of this new measurement is independent of the measurement errors of the old measurements. In such a case, the joint matrix $\mathbf{R}^{\{A0, B0, A1\}}$ is given by

$$\mathbf{R}^{\{A0, B0, A1\}} = \begin{bmatrix} \mathbf{R}^A & \mathbf{R}^{AB} & \mathbf{0} \\ (\mathbf{R}^{AB})^T & \mathbf{R}^B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}^A \end{bmatrix}, \quad (25)$$

where \mathbf{R}^{AB} is unknown. An upper bound $\mathbb{R}^{\{A0, B0, A1\}}$ of $\mathbf{R}^{\{A0, B0, A1\}}$ can be constructed as

$$\mathbb{R}^{\{A0, B0, A1\}}(\omega) = \begin{bmatrix} \frac{1}{\omega} \mathbf{R}^A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{1-\omega} \mathbf{R}^B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}^A \end{bmatrix}, \quad (26)$$

where $0 \leq \omega \leq 1$. Then, the joint weight $\mathbf{W}^{\{A0, B0, A1\}}$ that leads to the best upper bound $\mathbb{P}_{0,0;1}^{A,B;A}$ for a fixed value of the parameter ω is given by

$$\begin{aligned} \mathbf{W}^{\{A0, B0, A1\}}(\omega) &= ((1+\omega)(\mathbf{R}^A)^{-1} + (1-\omega)(\mathbf{R}^B)^{-1})^{-1} \\ &\cdot [\omega(\mathbf{R}^A)^{-1}, (1-\omega)(\mathbf{R}^B)^{-1}, (\mathbf{R}^A)^{-1}] = \\ &= \begin{bmatrix} \frac{2\omega}{3+\omega} & 0 & \frac{1-\omega}{3+\omega} & 0 & \frac{2}{3+\omega} & 0 \\ 0 & \frac{\omega}{3-\omega} & 0 & \frac{2(1-\omega)}{3-\omega} & 0 & \frac{1}{3-\omega} \end{bmatrix} \end{aligned} \quad (27)$$

and it holds

$$\begin{aligned} \mathbb{P}_{0,0;1}^{A,B;A}(\omega) &= ((1+\omega)(\mathbf{R}^A)^{-1} + (1-\omega)(\mathbf{R}^B)^{-1})^{-1} = \\ &= \begin{bmatrix} \frac{2}{3+\omega} & 0 \\ 0 & \frac{2}{3-\omega} \end{bmatrix}. \end{aligned} \quad (28)$$

The determinant and the trace of $\mathbb{P}_{0,0;1}^{A,B;A}(\omega)$ are minimised by $\omega = 0$. Because this is different from $\omega = \frac{1}{2}$, the ratio of the block components that correspond to $A0$ and $B0$ in $\mathbf{W}^{\{A0, B0, A1\}}$ is different from the ratio of \mathbf{W}_0^A and $\mathbf{I} - \mathbf{W}_0^A$. Thus, the optimal fusion cannot be achieved recursively.

The value $\omega = 0$ is expectable. Because the measurement errors of the measurements \mathcal{Z}_0^A and \mathcal{Z}_1^A have the same stochastic properties, it is advantageous to combine the measurement \mathcal{Z}_0^B with the measurement whose measurement error is independent of the measurement error of \mathcal{Z}_0^B . So, knowing that \mathcal{Z}_1^A will be available in the future, it is better to reject \mathcal{Z}_0^A in time $k = 0$ and wait for \mathcal{Z}_1^A than to do the fusion with \mathcal{Z}_0^A and proceed later with the fusion with \mathcal{Z}_1^A .

Exploiting the symmetry in (23), it is obvious that if the fusion of $\mathcal{X}_0^A, \mathcal{X}_0^B$ is done in filter B , the value $\omega = 1$ will be chosen. Therefore, the fusion under unknown dependence differs from the fusion under full knowledge. Since there is a free parameter ω that enters into the equations non-linearly, it does not generally hold that the best processing of the estimates with the best guaranteed quality lead to the estimates with the best guaranteed quality. Consequently, the same information can be fused differently, depending on the information that will be available in the future.

B. Dynamical system

This example focuses on the dependence of the optimal value of ω on the time κ . It shows that although the extreme values of ω can often be optimal, e.g. as in the previous example, they are not guaranteed to be optimal, neither asymptotically.

Let the linear stochastic system (1)–(3) be given by

$$\mathbf{F} = 0.95\mathbf{I}, \quad \mathbf{G} = \mathbf{I}, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0.95 \\ 0.95 & 1 \end{bmatrix}, \quad \mathbb{P}_0 = 10\mathbf{I}, \quad (29)$$

$$\mathbf{H}^A = [1 \ 0], \quad \mathbf{H}^B = [0 \ 1], \quad \mathbf{R}^A = 1, \quad \mathbf{R}^B = 3 \quad (30)$$

and let the local estimates $\hat{\chi}_0^A$, $\hat{\chi}_0^B$ be the minimum mean square error estimates. Take the exact mean square error matrices \mathbb{P}_k^A , \mathbb{P}_k^B as their upper bounds \mathbb{P}_k^A , \mathbb{P}_k^B .

The optimal recursive estimation that starts with the fused estimate is given by

$$\hat{\chi}_{k;\kappa}^{F;A} : \hat{\chi}_{k;\kappa}^{F;A} = (\mathbf{I} - \mathbf{K}_{k;\kappa}^{F;A} \mathbf{H}) \mathbf{F} \hat{\chi}_{k;\kappa-1}^{F;A} + \mathbf{K}_{k;\kappa}^{F;A} \mathcal{Z}_{k+\kappa}^A, \quad (31)$$

where $\hat{\chi}_{k;0}^{F;A} = \hat{\chi}_k^F$,

$$\mathbf{K}_{k;\kappa}^{F;A} = (\mathbf{F} \mathbb{P}_{k;\kappa-1}^{F;A} \mathbf{F}^T + \mathbf{G} \mathbf{Q} \mathbf{G}^T) (\mathbf{H}^A)^T \cdot (\mathbf{H}^A (\mathbf{F} \mathbb{P}_{k;\kappa-1}^{F;A} \mathbf{F}^T + \mathbf{G} \mathbf{Q} \mathbf{G}^T) (\mathbf{H}^A)^T + \mathbf{R}^A)^{-1}, \quad (32)$$

$$\mathbb{P}_{k;\kappa}^{F;A} = (\mathbf{I} - \mathbf{K}_{k;\kappa}^{F;A} \mathbf{H}) (\mathbf{F} \mathbb{P}_{k;\kappa-1}^{F;A} \mathbf{F}^T + \mathbf{G} \mathbf{Q} \mathbf{G}^T) \cdot (\mathbf{I} - \mathbf{K}_{k;\kappa}^{F;A} \mathbf{H})^T + \mathbf{K}_{k;\kappa}^{F;A} \mathbf{R}^A (\mathbf{K}_{k;\kappa}^{F;A})^T \quad (33)$$

and $\mathbb{P}_{k;0}^{F;A} = \mathbb{P}_k^F$ is used.

The batch approach uses nearly the same relations, the difference is again that the parameter ω is not fixed, but it is κ -dependent. So, $\mathbb{P}_{k,k;\kappa}^{A,B;A}(\omega, \mathbf{W}_k^A, \mathbf{K}_{k,k;\kappa}^{A,B;A})$ is optimised for each κ individually. Because the ω -optimal values $\mathbf{W}_k^A(\omega)$ and $\mathbf{K}_{k,k;\kappa}^{A,B;A}(\omega)$ follow from a completion to square for a given parameter ω , it remains to optimise $\mathbb{P}_{k,k;\kappa}^{A,B;A}$ only with respect to ω . In the optimal batch approach, the recursive relation (31) starts with $\hat{\chi}_{k,k;0}^{A,B;A}(\omega) = \hat{\chi}_k^F(\omega)$ and uses $\mathbf{K}_{k,k;\kappa}^{A,B;A}(\omega)$ from (32), while (33) starts with $\mathbb{P}_{k,k;0}^{A,B;A}(\omega) = \mathbb{P}_k^F(\omega)$ and the κ -dependent parameter ω is the one that optimises $\mathbb{P}_{k,k;\kappa}^{A,B;A}(\omega)$ over the domain of ω .

In this example, the fusion of the local estimates $\hat{\chi}_k^A$, $\hat{\chi}_k^B$ is considered at the time $k = 0$ and in the steady state, $k \rightarrow \infty$. The upper bound $\mathbb{P}_{k;\kappa}^{F;A}$ that is based on the fixed parameter ω is compared with the upper bound $\mathbb{P}_{k,k;\kappa}^{A,B;A}$ that uses a time variant parameter ω . The determinant and the trace are used for the comparison. Also, the upper bounds $\mathbb{P}_{k;\kappa}^{F;B}$ are compared with the upper bounds $\mathbb{P}_{k,k;\kappa}^{A,B;B}$.

At $k = 0$, the fused upper bound is given by

$$\mathbb{P}_0^F(\omega) = \begin{bmatrix} \frac{10}{1+10\omega} & 0 \\ 0 & \frac{30}{13-10\omega} \end{bmatrix} \quad (34)$$

and so the fusion is given by $\omega = 0.6$ if the criterion is the determinant of \mathbb{P}_0^F and by $\omega \approx 0.41$, where the symbol \approx means approximately equal, if the criterion is the trace of \mathbb{P}_0^F .

Fig. 3 shows the time evolution of the optimal value of ω , if the determinants and traces of $\mathbb{P}_{0,0;\kappa}^{A,B;A}$ and $\mathbb{P}_{0,0;\kappa}^{A,B;B}$ are minimised. It can be observed that for $\kappa \rightarrow \infty$, the optimal values approach some constant values. These values are the same for the criterion given by the determinant and by the trace, but differ for different future data. In the filter A , the steady optimal value is $\omega \approx 0.04$. That means that although it is optimal to combine $\hat{\chi}_0^A$ with $\hat{\chi}_0^B$ by comparable weights at $k = 0$, the filter A may decide to almost ignore its own estimate $\hat{\chi}_0^A$, because the measurements \mathcal{Z}_κ^A will lead to estimates $\hat{\chi}_{0,0;\kappa}^{A,B;A}$ with better upper bounds of the mean square error matrices in the future. In filter B , the steady optimal value is $\omega \approx 0.85$. Similarly to the filter A , the filter B puts a small weight on its own estimate $\hat{\chi}_0^B$, if a sufficient number of measurements \mathcal{Z}_κ^B is to come in the future.

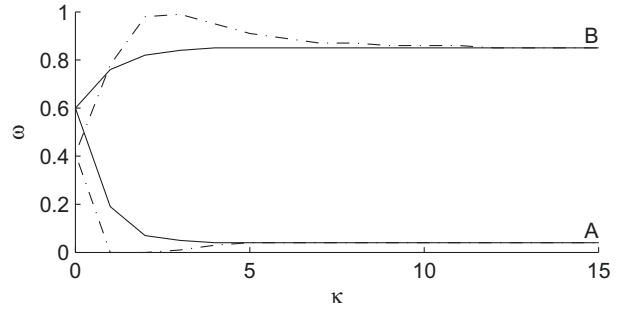


Fig. 3. The optimal value of ω as a function of time κ for the fusion made at $k = 0$. The optimisation of determinant in filters A and B (solid lines), the optimisation of trace in filters A and B (dash-dotted lines).

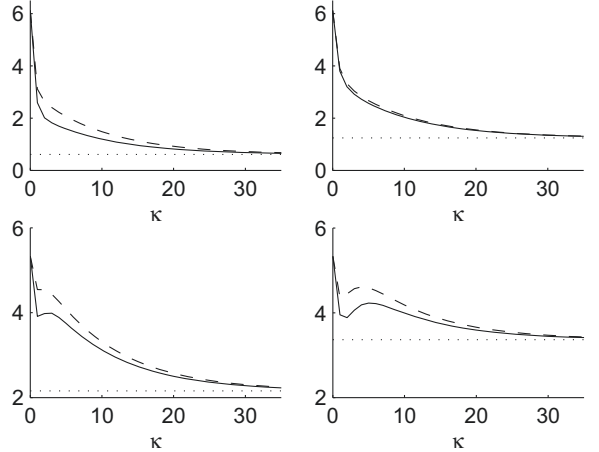


Fig. 4. Upper bounds of the mean square error matrices as functions of time κ for the fusion made at $k = 0$. Determinants – upper figures, traces – lower figures, filter A – left figures, filter B – right figures, parameter ω fixed at the fusion time k – dashed lines, parameter ω optimised for each κ – solid lines, steady state values – dotted lines.

Fig. 4 compares the minimal values of determinants and traces of $\mathbb{P}_{0,0;\kappa}^{A,B;A}$ and $\mathbb{P}_{0,0;\kappa}^{A,B;B}$ with the values of determinant and traces of $\mathbb{P}_{0;\kappa}^{F;A}$ and $\mathbb{P}_{0;\kappa}^{F;B}$, i.e. of the upper bounds of the mean square error matrices of the estimators that are based on the fixed parameter ω . It is evident that for $\kappa \rightarrow \infty$, the influence of the choice of ω on the upper bounds vanishes. In the steady state, it holds $\mathbb{P}_{k,k;\infty}^{A,B;A} = \mathbb{P}_\infty^A$ and $\mathbb{P}_{k,k;\infty}^{A,B;B} = \mathbb{P}_\infty^B$, where the steady state matrices are given by

$$\mathbb{P}_\infty^A \approx \begin{bmatrix} 0.6076 & 0.5772 \\ 0.5772 & 1.5483 \end{bmatrix}, \quad \mathbb{P}_\infty^B \approx \begin{bmatrix} 2.1216 & 1.1806 \\ 1.1806 & 1.2427 \end{bmatrix} \quad (35)$$

and are obtained from Riccati equations. Also, the difference between the optimal and the fixed ω solution is small, if the optimal ω is close to the fixed value ω , see the determinant criterion in filter B .

Fig. 5 visualises the upper bound matrices. For two dimensional vector $\tilde{\mathbf{x}}$, $\tilde{\mathbf{x}} = [\tilde{x}_1, \tilde{x}_2]^T$, and \mathbb{P}^{-1} representing an upper bound, the set $\{\tilde{\mathbf{x}} : \tilde{\mathbf{x}}^T \mathbb{P}^{-1} \tilde{\mathbf{x}} = 1\}$ forms an ellipse. The ellipses corresponding to local and fused upper bounds are shown, where the fused upper bounds \mathbb{P}_0^F use the optimal values of ω for $\kappa = 0$ with determinant and trace criterion and for $\kappa \rightarrow \infty$ at filters A and B , see also Fig. 3. As it was described above, filter A prefers \mathbb{P}_0^F that is close to \mathbb{P}_0^B , because the measurements \mathcal{Z}_κ^A will be squeezing the ellipse

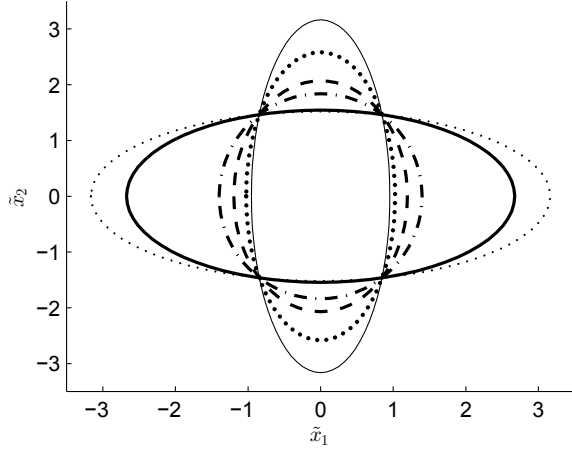


Fig. 5. Ellipses given by $\tilde{\mathbf{x}}^T \mathbb{P}^{-1} \tilde{\mathbf{x}} = 1$, fusion at $k = 0$. Thin ellipses – local bound \mathbb{P}_0^A (solid), local bound \mathbb{P}_0^B (dotted). Bold ellipses – fused bounds \mathbb{P}_0^F optimal with respect to determinant (dashed), trace (dash-dotted) and both measures in the steady state at the filter A (solid) and at the filter B (dotted).

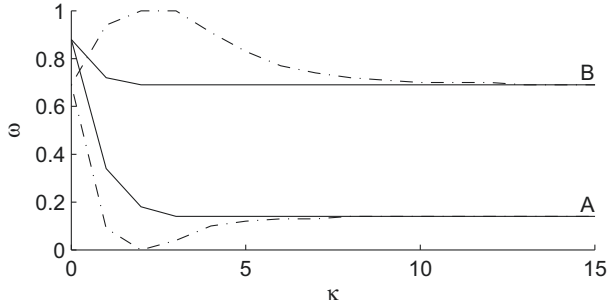


Fig. 6. The optimal value of ω as functions of time κ for the fusion made at the steady state, $k \rightarrow \infty$. The optimisation of determinant in filters A and B (solid lines), the optimisation of trace in filters A and B (dash-dotted lines).

in the \tilde{x}_1 direction. Similarly, filter B prefers \mathbb{P}_0^F that is close to \mathbb{P}_0^A , because the measurements \mathcal{Z}_κ^B will be squeezing the ellipse in the \tilde{x}_2 direction. The parameters ω optimal with respect to $\kappa = 0$ lead to ellipses that are close to a circle, because no further measurements are considered.

At $k \rightarrow \infty$, the fusion is given by $\omega \approx 0.88$ and $\omega \approx 0.69$, if the criterion is the determinant and trace respectively of \mathbb{P}_∞^F . Now, the fusion clearly prefers the estimate $\hat{\mathcal{X}}_\infty^A$ over the estimate $\hat{\mathcal{X}}_\infty^B$.

Fig. 6 shows the time evolution of the optimal value of ω , if the determinants and traces of $\mathbb{P}_{\infty, \infty; \kappa}^{A, B; A}$ and $\mathbb{P}_{\infty, \infty; \kappa}^{A, B; B}$ are minimised, while the upper bounds are given as limits, $\mathbb{P}_{\infty, \infty; \kappa}^{A, B; A} = \lim_{k \rightarrow \infty} \mathbb{P}_{k, k; \kappa}^{A, B; A}$, $\mathbb{P}_{\infty, \infty; \kappa}^{A, B; B} = \lim_{k \rightarrow \infty} \mathbb{P}_{k, k; \kappa}^{A, B; B}$. Again, for $\kappa \rightarrow \infty$, the optimal values approach some constant values, that differ for different future data but do not differ for the determinant and the criterion. The steady optimal values is now given by $\omega \approx 0.14$ in the filter A and $\omega \approx 0.69$ in the filter B . Comparing to the case $k = 0$, the fusion at a filter puts larger weight on its own estimate, if the local measurements are expected to be available.

Fig. 7 compares the minimal values of determinants and traces of $\mathbb{P}_{\infty, \infty; \kappa}^{A, B; A}$ and $\mathbb{P}_{\infty, \infty; \kappa}^{A, B; B}$ with the values of determinant and traces of $\mathbb{P}_{\infty, \infty; \kappa}^{F; A}$ and $\mathbb{P}_{\infty, \infty; \kappa}^{F; B}$. Again, the influence of the

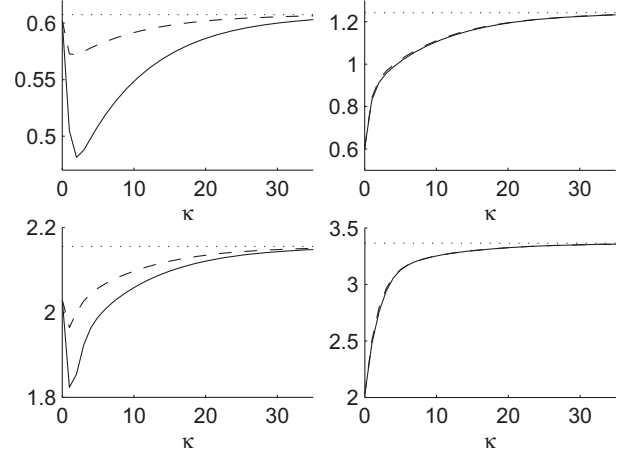


Fig. 7. Upper bounds of the mean square error matrices as a function of time κ for the fusion made at the steady state, $k \rightarrow \infty$. Determinants – upper figures, traces – lower figures, filter A – left figures, filter B – right figures, parameter ω fixed at the fusion time k – dashed lines, parameter ω optimised for each κ – solid lines, steady state values – dotted lines.

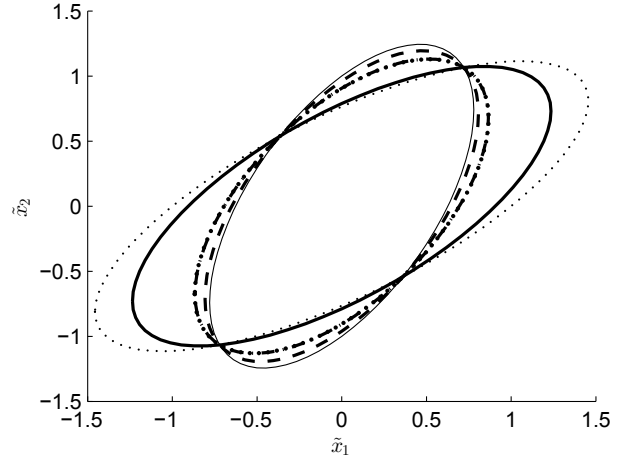


Fig. 8. Ellipses given by $\tilde{\mathbf{x}}^T \mathbb{P}^{-1} \tilde{\mathbf{x}} = 1$, fusion at the steady state, $k \rightarrow \infty$. Thin ellipses – local bound \mathbb{P}_∞^A (solid), local bound \mathbb{P}_∞^B (dotted). Bold ellipses – fused bounds \mathbb{P}_∞^F optimal with respect to determinant (dashed), trace (dash-dotted) and both measures in the steady state at the filter A (solid) and at the filter B (dotted, aliased with the bold dash-dotted ellipse).

choice of ω on the upper bounds vanishes for $\kappa \rightarrow \infty$. At the filter B , the difference between the optimal and the fixed ω solution is negligible, since the optimal ω is close to the fixed value ω for both the determinant and trace criterion.

Fig. 8 visualises the upper bound matrices. The ellipses corresponding to local and fused upper bounds are shown, where the fused upper bounds \mathbb{P}_∞^F use the optimal values of ω for $\kappa \rightarrow \infty$ with determinant and trace criterion and for $\kappa \rightarrow \infty$ at filters A and B , see also Fig. 6. An observation similar to that at Fig. 5 can be made here.

V. CONCLUSION

The paper has considered the connection of linear fusion under unknown cross-correlations with the state estimation of linear dynamical system. It has been shown that the estimator with the minimal upper bound of the mean square error matrix cannot be obtained recursively. The optimal parameter of the

upper bound is dependent on the amount of information that will be available after the fusion is performed. Thus, two filters with different sources of measurement can fuse the same information differently. The best representation of ignorance do depend on the way the future measurements are obtained.

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