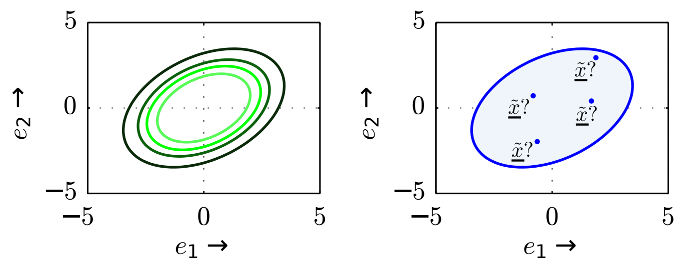
Data Validation in the Presence of Stochastic and Set-membership Uncertainties

Florian Pfaff, **Benjamin Noack**, and **Uwe D. Hanebeck**
Intelligent Sensor-Actuator-Systems Laboratory (ISAS)
Institute for Anthropomatics
Karlsruhe Institute of Technology (KIT), Germany
florian.pfaff@student.kit.edu, benjamin.noack@ieee.org, uwe.hanebeck@ieee.org

***Abstract*—For systems suffering from different types of uncertainties, finding criteria for validating measurements can be challenging. In this paper, we regard both stochastic Gaussian noise with full or imprecise knowledge about correlations and unknown but bounded errors. The validation problems arising in the individual and combined cases are illustrated to convey different perspectives on the proposed conditions. Furthermore, hints are provided for the algorithmic implementation of the validation tests. Particular focus is put on ensuring a predefined lower bound for the probability of correctly classifying valid data.**

Index Terms*—*data validation, imprecisely known correlations, set-membership uncertainties, unknown but bounded errors



(a) Multiple scaled covariance ellipsoids for a bivariate Gaussian distribution.

(b) Ellipsoidal set containing multiple values, one of which is the true parameter $\tilde{\underline{x}}$.

Figure 1. Comparison of stochastic and set-membership uncertainty.

I. Introduction

In data validation, we are concerned with deciding whether new data, usually in form of a measurement, appears to be valid under specified criteria. Commonly, the main concern is if the estimate stemming from the new data is sufficiently compatible with the current estimate from a statistical point of view. For example, in the case of Gaussian distributions and an underlying linear relationship, the compatibility can be tested using the Mahalanobis distance [1].

More generally, we can employ hypotheses tests [2] to perform the judgment. A hypothesis test is a prespecified rule determining when to accept or reject hypotheses (statements about distributions) based on the data observed. Our null hypothesis is that the measurement is valid and thus compatible with our current estimate. Our aim throughout the paper is to ensure that less than α, e.g. 5%, of actually compatible estimates are expected to result in rejection of the measurement. As common in hypothesis testing, we refer to α as the type I error probability.

In the easiest case, we can express our hypothesis as a fully specified distribution. We can then divide its possible realizations into an acceptance and a rejection region. As the acceptance region, we select a subset of realizations that covers (at least) $1 - \alpha$ of probability mass when conditioned on the null hypotheses. However, even in the linear case with Gaussian noise, the null hypothesis cannot be formulated as a fully specified distributions if the correlations are not fully known. In this case, the hypothesis has to respect multiple possible covariances as possible parameters. The approach taken in [3] generalizes the aforementioned test using the Mahalanobis distance by classifying measurements as valid if they appear to be compatible under any of the correlations that are considered to be possible. If we regard the resulting acceptance region, this criterion tests for containment in the union of the individual acceptance regions. Tests constructed this way are known in Neyman–Pearson hypothesis testing as intersection–union tests [2]. While significantly differing tests could be thought of, this test successfully ensures that we attain a type I error probability of less than α for our composite hypothesis [4].

Aside from requiring knowledge about correlations, another prerequisite for formulating a hypothesis is obviously basic knowledge about the possible distributions of the errors. For errors present in a variety of sensors (e.g., image noise [5]), stochastic models can be built that are suitable for approximating the noise behavior under certain circumstances.

However, there are perturbations for which assuming a distribution may not make sense. Due to engineering tolerances or inaccurate calibration, devices can suffer from constant biases, whose bounds may be known from the devices' specifications. A variable source of error, which is bounded but generally no distribution is known of, is noise stemming from discretization. In these and other cases, unknown but bounded errors cause us to consider several elements, intervals, or other infinite sets of possible values instead of an actual distribution. Therefore, we also refer to them as set-membership errors. Evidently, it is also possible that the uncertainties split into two parts: one of which we know the distribution of and one of which we can only tell the set the true parameter is a member of. We have illustrated the difference between the two types of uncertainties by putting a series of scaled covariance ellipsoids of the same distribution side by side to an ellipsoidal set of similar shape in Fig. 1.

In the process of this paper, we lay out the challenges for data validation in the presence of the aforementioned uncertainties and suggest validity tests with a type I error probability of less than α when stochastic and set-membership uncertainties and are present, additionally considering imprecisely known correlations of the former. The concepts can be regarded in a set of densities framework that is already in use for several filtering algorithms [6], [7], [8], with extensions available in the ellipsoidal case [9], [10]. Due to useful properties and prevalence in filtering algorithms, only ellipsoidal sets are covered in this paper.

The next section deals with inspecting the various cases and visualizes their challenges. Methods for numerically solving the data validation problems are addressed in section III. After presenting an example and results of experiments for the data validation for both stochastic and set-membership uncertainties, we conclude the paper in the last section with a summary and an outlook.

II. Explanation of the Various Cases

In this section, we walk through the various cases, visually explaining the difficulties arising. First, we begin with the case of known Gaussian distributions and known correlations. Then, we proceed with the validation in the presence of unknown but bounded errors. The third subsection deals with Gaussian noise and imprecisely known correlations. In the fourth and fifth subsection, the set-membership uncertainties are first combined with Gaussian noise under known and then with Gaussian noise under imprecisely known correlations.

All sets used for the set-membership uncertainties are assumed to be ellipsoidal. $\mathcal{E}(\underline{q}, \mathbf{Q})$ denotes the ellipsoid described by the centroid $\underline{q}$ and the shape matrix $\mathbf{Q}$. Please be aware that multiple interpretations exist in the literature and we use the definition

$$\underline{a} \in \mathcal{E}(\underline{q}, \mathbf{Q}) \Leftrightarrow (\underline{a} - \underline{q})^T \mathbf{Q}^{-1} (\underline{a} - \underline{q}) \leq 1 \ ,$$

which involves inverting the shape matrix. By underlining constants or variables we indicate that they are usually vectors.

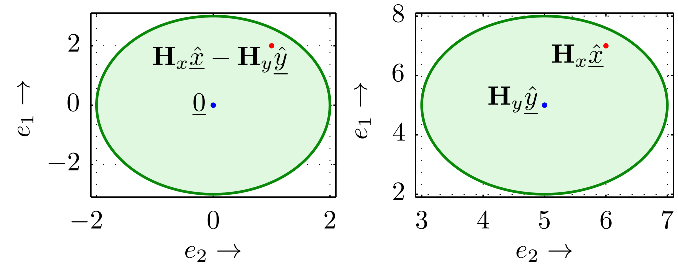
For testing the compatibility, we have to put two estimates in relation. We assume two linear mappings $\mathbf{H}_x$ and $\mathbf{H}_y$ exist that relate the true value of $\underline{x}$, written as $\underline{\tilde{x}}$ to the true value of $\underline{y}$, written as $\underline{\tilde{y}}$, in the form of

$$\mathbf{H}_x \underline{\tilde{x}} = \mathbf{H}_y \underline{\tilde{y}} \ .$$

We focus on criteria that ensure at least a probability of $1-\alpha$ for not committing a type I error. By keeping the acceptance regions small, we can potentially reduce the probability of misclassifying invalid measurements (type II error probability). However, minimization of the type II error probability is not attempted. In fact, as we leave the alternative hypothesis open to be specified on a case-by-case basis, we are unable to determine specific probabilities for type II errors.

A. Stochastic Error with Known Correlations Only

For Gaussian distributions, the acceptance regions yielding a probability of exactly $1-\alpha$ for correctly classifying valid data that contain only the observations most likely under the null hypothesis are always scaled covariance ellipsoids.



(a) Covariance ellipsoid of $\underline{\boldsymbol{z}}$.

(b) Covariance ellipsoid drawn around $\mathbf{H}_y\underline{\hat{y}}$.

Figure 2. Illustration of equivalent tests for the containment in the covariance ellipsoid.

These bounds are well known and commonly used, such as in the empirical rule. We call scaled covariance ellipsoids $k\mathbf{C}$-bounds for $k \in \mathbb{R}^+$—k can be chosen to ensure a type I error probability of less than α. This is slightly different from the notation of the "g-sigma ellipsoid" used in [11], which involves a similar parameter g in squared form.

For verifying the compatibility, we define the new random variable (denoted by a bold letter)

$$\underline{\boldsymbol{z}} = \mathbf{H}_x \underline{\boldsymbol{x}} - \mathbf{H}_y \underline{\boldsymbol{y}}$$

and test if the current estimate $\underline{\hat{z}}$, calculated from the individual estimates $\underline{\hat{x}}$ and $\underline{\hat{y}}$, is contained in the $k\mathbf{C}$-bound of the estimator that has zero mean and is distributed as $\underline{\boldsymbol{z}} \sim \mathcal{N}(\underline{0}, \mathbf{C}_z)$. As the correlations are known, $\mathbf{C}_z$ can be calculated exactly and the condition, as illustrated in Fig. 2(a), becomes

$$\underline{\hat{z}} = \mathbf{H}_x \underline{\hat{x}} - \mathbf{H}_y \underline{\hat{y}} \in \mathcal{E}(\underline{0}, k\mathbf{C}_z) \ , \qquad (1)$$

yielding $\mathcal{E}(\underline{0}, k\mathbf{C}_z)$ as our acceptance region.

Considering this definition, we can infer that another intuitive condition shown in Fig. 2(b) can be used for validation instead. The two estimates $\underline{\hat{x}}$ and $\underline{\hat{y}}$ are deemed compatible if drawing the $k\mathbf{C}_z$-bound around $\mathbf{H}_x\underline{\hat{x}}$ will encompass $\mathbf{H}_y\underline{\hat{y}}$ or, equivalently, drawing the $k\mathbf{C}_z$-bound around $\mathbf{H}_y\underline{\hat{y}}$ contains $\mathbf{H}_x\underline{\hat{x}}$. In other words, we test if

$$\mathbf{H}_x \underline{\hat{x}} \in \mathcal{E}(\mathbf{H}_y \underline{\hat{y}}, k\mathbf{C}_z) \qquad (2)$$

holds.

B. Set-membership Error Only

In so called Guaranteed State Estimation [12], [13], unknown but bounded errors are dealt with. In this case, the estimates form sets $\mathcal{X}$ and $\mathcal{Y}$ that satisfy $\underline{\tilde{x}} \in \mathcal{X}$ and $\underline{\tilde{y}} \in \mathcal{Y}$ at each time step. The new estimate is usually calculated by determining an intersection [14]. In this context, the current estimate—often the set of possible true parameters—is often referred to as the information, consistency, or feasibility domain [13]. If both $\mathcal{X}$ and $\mathcal{Y}$ are valid, the intersection is never empty. Crucially, if one or both of the sets are incorrect, meaning they do not contain the respective true values, the intersection can be empty. Performing an update step under this condition can lead to the empty set being the nonsensical, new estimate.

By using data validation, we try to avoid the above problem. In our case, we have two ellipsoids $\mathbf{H}_x\mathcal{X}$ and $\mathbf{H}_y\mathcal{Y}$ (Fig. 3)

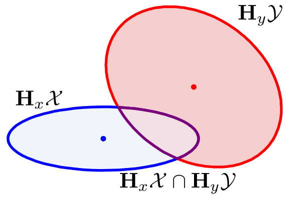


Figure 3. Transformed ellipsoids having common points.

with $\underline{\tilde{x}} \in \mathcal{X}$ and $\underline{\tilde{y}} \in \mathcal{Y}$—if both are valid. As we consider any pair $(\underline{x}, \underline{y}) \in \mathcal{X} \times \mathcal{Y}$, we have to accept the measurement if

$$\mathbf{H}_x\mathcal{X} \cap \mathbf{H}_y\mathcal{Y} \neq \emptyset \ .$$

Otherwise, it is possible that $(\underline{\tilde{x}}, \underline{\tilde{y}}) \in \mathcal{X} \times \mathcal{Y}$, leading to the rejection of a valid measurement. As the only property known about a valid estimate is that it contains the real value, we have no way to distinguish between a valid measurement and an invalid measurement that results in a nonempty intersection. However, we can safely reject all $\mathcal{Y}$ for which the intersection is empty as we do not consider an additional stochastic error.

C. Stochastic Error with Imprecisely Known Correlations Only

When correlations are imprecisely known, there are usually multiple possible covariance matrices $\mathbf{C}_z$. If we call the set of correlations that are allowed—based both on our imprecise knowledge and on if they describe valid correlations—$\mathcal{R}$, then we get a family of covariances $\mathbf{C}_z(\underline{r})$ with $\underline{r} \in \mathcal{R}$. In order to achieve a type I error probability of less than α, we demand that the condition from subsection II-A is satisfied for at least one of the possible correlations by determining if

$$\exists \underline{r} \in \mathcal{R} : \mathbf{H}_x\underline{\hat{x}} - \mathbf{H}_y\underline{\hat{y}} \in \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r}))$$

holds. This condition is equivalent to

$$\mathbf{H}_x\underline{\hat{x}} - \mathbf{H}_y\underline{\hat{y}} \in \bigcup_{\underline{r} \in \mathcal{R}} \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r})) \ ,$$

which is basically testing for containment in the union of the acceptance regions. The area resulting from taking the union of multiple ellipsoids is demonstrated in Fig. 4(a) and is not an ellipsoid in general. As testing for containment in the union of these covariance ellipsoids is generally difficult, we take another condition into consideration. In subsection III-C, we introduce a family of ellipsoids with shape matrices $\mathbf{V}(\kappa)$ whose intersection is equivalent to the union above, meaning

$$\bigcup_{\underline{r} \in \mathcal{R}} \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r})) = \bigcap_{\kappa} \mathcal{E}(\underline{0}, k\mathbf{V}(\kappa)) \ . \tag{3}$$

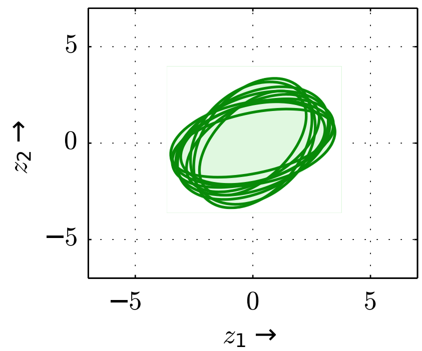
This allows us to perform the test in a different fashion using the condition

$$\mathbf{H}_x\underline{\hat{x}} - \mathbf{H}_y\underline{\hat{y}} \in \bigcap_{\kappa} \mathcal{E}(\underline{0}, k\mathbf{V}(\kappa)) \ ,$$

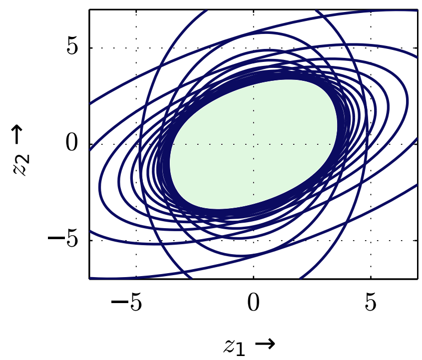
which is shown in Fig. 4(b).

Remark 1. The union of the acceptance regions may not be the smallest acceptance region—it can actually be quite conservative [4]. In many cases, the probability of not committing a type I error is strictly above $1-\alpha$ for the above described method. Imagine $\underline{\tilde{r}}$ being the true (and known) correlations, then the probability of not committing a type I error is

$$P(\mathbf{H}_x\underline{\hat{x}} - \mathbf{H}_y\underline{\hat{y}} \in \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{\tilde{r}}))| \ \underline{\hat{y}} \text{ valid}) = 1-\alpha$$



(a) Some of the possible $\mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r}))$ for $k=1$.



(b) Example for bounds in the family $\mathcal{E}(\underline{0}, k\mathbf{V}(\kappa))$ for $k=1$.

Figure 4. Example for ellipsoids with shape matrices $\mathbf{C}_z(\underline{r})$ and $\mathbf{V}(\kappa)$ in the case of two-dimensional random vectors with symmetrically constrained bounds for the maximum absolute correlation.

for the single $k\mathbf{C}(\underline{\tilde{r}})$-bound. If there is a family of possible correlations $\mathcal{R}$, then we get

$$P\big(\mathbf{H}_x\underline{\hat{x}} - \mathbf{H}_y\underline{\hat{y}} \in \bigcup_{\underline{r} \in \mathcal{R}} \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r}))| \ \underline{\hat{y}} \text{ valid}\big) \geq 1-\alpha$$

for the union of the $k\mathbf{C}(\underline{r})$-bounds, regardless which of these $\underline{r}$ is considered. By "$\underline{\hat{y}}$ valid", we mean that $\underline{\hat{y}}$ is only perturbed by a noise of a distribution assumed to be possible in the respective case. For visualization purposes, you can imagine that the union of all possible $k\mathbf{C}(\underline{r})$-bound ellipsoids includes the $k\mathbf{C}(\underline{r})$-bound ellipsoid for every individual density plus a part beyond which covers additional probability mass.

As finding tighter bounds can only potentially reduce the type II error probability which we do not specifically aim to minimize, this issue is not dealt with in this paper. Similar issues arise in the following subsections but are not addressed any further.

D. Stochastic Error with Known Correlations and Set-membership Error

Now we assume that the error consists of a stochastic part of which the correlations are known and a set-membership part. The stochastic uncertainty is described by a Gaussian distribution with covariances $\mathbf{C}_{xx}$ and $\mathbf{C}_{yy}$, while the unknown but bounded errors are described by ellipsoids $\mathcal{X} = \mathcal{E}(\underline{\hat{x}}, \mathbf{X})$ and $\mathcal{Y} = \mathcal{E}(\underline{\hat{y}}, \mathbf{Y})$. We regard all elements of $\mathcal{X}$ and $\mathcal{Y}$ as possible means of Gaussian distributions with covariances $\mathbf{C}_{xx}$ and $\mathbf{C}_{yy}$, respectively.

To ensure a type I error probability of less than α, we test if the condition (2) is satisfied for any pair $(\underline{x}, \underline{y}) \in \mathcal{X} \times \mathcal{Y}$ using the condition

$$\exists \underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \mathbf{H}_x \underline{x} \in \mathcal{E}(\mathbf{H}_y \underline{y}, k\mathbf{C}_z) \ . \tag{4}$$

Visually speaking, we are taking the $k\mathbf{C}_z$-bound around every $\mathbf{H}_y \underline{y}$ with $\underline{y} \in \mathcal{Y}$ and test if $\mathbf{H}_x \underline{x}$ is contained for any $\underline{x} \in \mathcal{X}$. This is demonstrated in Fig. 5(a). Alternatively, we could also do this the other way around.

Drawing the covariance ellipsoid around every $\mathbf{H}_y \underline{y}$ basically results in the Minkowski sum of $\mathcal{E}(\mathbf{H}_y \hat{\underline{y}}, \mathbf{H}_y \mathbf{Y} \mathbf{H}_y^T)$ and $\mathcal{E}(\underline{0}, k\mathbf{C}_z)$ (Fig. 5(b)). As this allows us to equivalently test if the intersection of $\mathbf{H}_x \mathcal{X}$ with the Minkowski sum is nonempty, our new validation criterion becomes

$$\mathcal{E}(\mathbf{H}_x \hat{\underline{x}}, \mathbf{H}_x \mathbf{X} \mathbf{H}_x^T) \cap (\mathcal{E}(\mathbf{H}_y \hat{\underline{y}}, \mathbf{H}_y \mathbf{Y} \mathbf{H}_y^T) \oplus \mathcal{E}(\underline{0}, k\mathbf{C}_z)) \neq \emptyset \ .$$

And this again is equivalent to testing if further adding the shape, but not the translation, of $\mathbf{H}_x \mathcal{X}$ to the Minkowski sum causes the sum to contain $\mathbf{H}_y \hat{\underline{x}}$. The condition

$$\mathbf{H}_x \hat{\underline{x}} \in \mathcal{E}(\underline{0}, \mathbf{H}_x \mathbf{X} \mathbf{H}_x^T) \oplus \mathcal{E}(\mathbf{H}_y \hat{\underline{y}}, \mathbf{H}_y \mathbf{Y} \mathbf{H}_y^T) \oplus \mathcal{E}(\underline{0}, k\mathbf{C}_z)$$

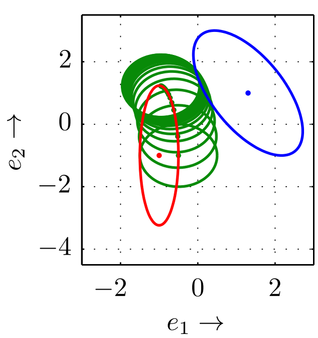
is illustrated in Fig. 5(c). Please note that this can also be written as

$$\mathbf{H}_x \hat{\underline{x}} - \mathbf{H}_y \hat{\underline{y}} \in \mathcal{E}(\underline{0}, \mathbf{H}_x \mathbf{X} \mathbf{H}_x^T) \oplus \mathcal{E}(\underline{0}, \mathbf{H}_y \mathbf{Y} \mathbf{H}_y^T) \oplus \mathcal{E}(\underline{0}, k\mathbf{C}_z) \ .$$

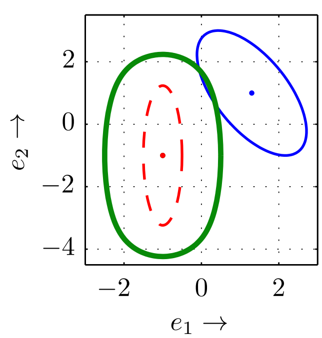
We could have arrived at this form faster, albeit less graphically, had we started with generalizing the condition (1) from subsection II-A as

$$\exists \underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \mathbf{H}_x \underline{x} - \mathbf{H}_y \underline{y} \in \mathcal{E}(\underline{0}, k\mathbf{C}_z)$$

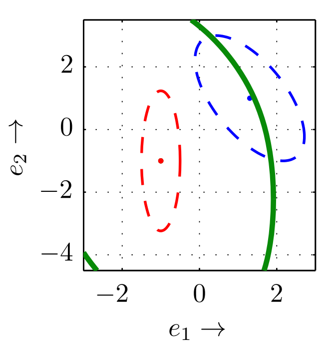
instead.



(a) $1\mathbf{C}$-bound ellipsoids drawn in dark green around some $\mathbf{H}_y \underline{y}$.



(b) $\mathbf{H}\mathcal{Y} \oplus \mathcal{E}(\underline{0}, \mathbf{C}_z)$ in dark green.



(c) Minkowski sum of all three in dark green.

Figure 5. Different ways of visually interpreting the condition for validity when $k = 1$.

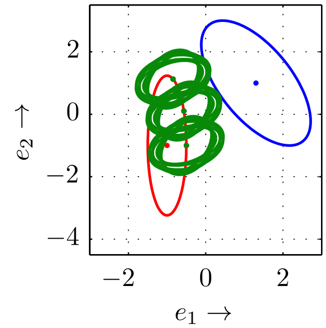


Figure 6. Visual representation of condition (5) similar to Fig. 5(a).

E. Stochastic Error with Imprecisely Known Correlations and Set-membership Error

While considering stochastic and set-membership errors as in the previous section, we now also respect all possible covariance matrices as done in subsection II-C. We demand the probability for a type I error to be less than α, regardless which of the possible correlations and which of the elements in $\mathcal{X}$ and $\mathcal{Y}$ are considered. In the visualization, Fig. 5(a) turns into Fig. 6 as it suffices if the condition is satisfied for any of the covariance matrices. Thus condition (4) becomes

$$\exists \underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y}, \exists \underline{r} \in \mathcal{R} : \mathbf{H}_x \underline{x} \in \mathcal{E}(\mathbf{H}_y \underline{y}, k\mathbf{C}_z(\underline{r})) \ . \tag{5}$$

As done similarly in subsection II-C we can equivalently test if

$$\exists \underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \mathbf{H}_x \underline{x} \in \bigcup_{\underline{r} \in \mathcal{R}} \mathcal{E}(\mathbf{H}_y \underline{y}, k\mathbf{C}_z(\underline{r})) \tag{6}$$

holds. From this form we only need two further steps for getting to the form we need in subsection III-E. In the first step, we write the above condition as

$$\exists \underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \mathbf{H}_x \underline{x} - \mathbf{H}_y \underline{y} \in \bigcup_{\underline{r} \in \mathcal{R}} \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r})) \ .$$

If we now—as mentioned in subsection II-C—consider a family of ellipsoids with shape matrices $\mathbf{V}(\kappa)$ whose intersection is equal to the union of the ellipsoids with shape matrices $\mathbf{C}_z(\underline{r})$, we can write the condition as

$$\exists \underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \mathbf{H}_x \underline{x} - \mathbf{H}_y \underline{y} \in \bigcap_{\kappa} \mathcal{E}(\underline{0}, k\mathbf{V}(\underline{r})) \ . \tag{7}$$

In order to get toward another helpful formulation, we consider that

$$\bigcup_{\underline{r} \in \mathcal{R}} \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r}))$$

is centrally symmetric, enabling us to write the condition (6) as

$$\exists \underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \mathbf{H}_y \underline{y} \in \bigcup_{\underline{r} \in \mathcal{R}} \mathcal{E}(\mathbf{H}_x \underline{x}, k\mathbf{C}_z(\underline{r}))$$

instead. This is not much of a surprise as the initial considerations could have led us to start with $\mathbf{H}_x \underline{x}$ and $\mathbf{H}_y \underline{y}$ swapped in condition (5). Following another step of subsection II-D and writing $\mathcal{E}(\mathbf{H}_x \hat{\underline{x}}, \mathbf{H}_x \mathbf{X} \mathbf{H}_x^T)$ for additional brevity as $\mathbf{H}_x \mathcal{X}$ yields the condition

$$\mathbf{H}_y \mathcal{Y} \cap (\mathbf{H}_x \mathcal{X} \oplus \bigcup_{\underline{r} \in \mathcal{R}} \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r}))) \neq \emptyset \ , \tag{8}$$

which is useful for intuitively interpreting the example in section IV.

III. Methods for Validation

Having shown all cases of interest, we now suggest ways to actually perform the validation. We assume that our knowledge of the imprecisely known correlations can be written as a bound for the maximum absolute correlation that bounds the total correlation coefficient r_{xy} by the inequality

$$|r_{xy}| \leq r_{max} \ .$$

For brevity, we interpret a symmetric, positive definite matrix $\mathbf{W}$ as a Gramian matrix of an inner product and use $d_\mathbf{W}$ for the induced metric. This allows us to write

$$\begin{aligned}&(\mathbf{H}_x\underline{x} - \mathbf{H}_y\underline{y})^T\mathbf{W}(\mathbf{H}_x\underline{x} - \mathbf{H}_y\underline{y})\\ =&\langle(\mathbf{H}_x\underline{x} - \mathbf{H}_y\underline{y}), (\mathbf{H}_x\underline{x} - \mathbf{H}_y\underline{y})\rangle_\mathbf{W} = d^2_\mathbf{W}(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \ .\end{aligned}$$

A. Stochastic Error with Known Correlations Only

If the correlations are known, we are therefore able to calculate $\mathbf{C}_{xy}$ and thus also $\mathbf{C}_z$ as

$$\begin{aligned}\mathbf{C}_z = \mathrm{Cov}\{\mathbf{H}_x\underline{\boldsymbol{x}} - \mathbf{H}_y\underline{\boldsymbol{y}}\} =&\mathbf{H}_x\mathbf{C}_{xx}\mathbf{H}_x^T + \mathbf{H}_y\mathbf{C}_{yy}\mathbf{H}_y^T\\ &-\mathbf{H}_x\mathbf{C}_{xy}\mathbf{H}_y^T - \mathbf{H}_y\mathbf{C}_{yx}\mathbf{H}_x^T \ .\end{aligned}$$

The condition (1) from subsection II-A can be verified in the usual manner for testing containment in an ellipsoid by testing the inequality

$$\begin{aligned}&\mathbf{H}_x\hat{\underline{x}} - \mathbf{H}_y\hat{\underline{y}} \in \mathcal{E}(\underline{0}, k\mathbf{C}_z)\\ \Leftrightarrow&(\mathbf{H}_x\hat{\underline{x}} - \mathbf{H}_y\hat{\underline{y}})^T(k\mathbf{C}_z)^{-1}(\mathbf{H}_x\hat{\underline{x}} - \mathbf{H}_y\hat{\underline{y}}) \leq 1 \ .\end{aligned}$$

By multiplying with k on both sides and setting $\mathbf{W} = \mathbf{C}_z^{-1}$, we can formulate the condition as

$$d^2_\mathbf{W}(\mathbf{H}_x\hat{\underline{x}}, \mathbf{H}_y\hat{\underline{y}}) \leq k \ ,$$

which is basically performing the test via the Mahalanobis distance.

B. Set-membership Error Only

Essentially only

$$\mathbf{H}_x\mathcal{X} \cap \mathbf{H}_y\mathcal{Y} \neq \emptyset$$

has to be tested. As the transformations can easily be calculated as

$$\mathbf{H}_x\mathcal{X} = \mathcal{E}(\mathbf{H}_x\hat{\underline{x}}, \mathbf{H}_x\mathbf{X}\mathbf{H}_x^T) \ ,$$

the problem is reduced to testing if two arbitrary-dimensional ellipsoids intersect.

A naive way of testing this criterion is by determining if their distance is zero. While we use distance calculation in a different subsection, the intersection test for two ellipsoids is a common problem, which has attracted several researchers to propose more efficient algorithms [15], [16]. It is still an active field of research with a recently released algorithm solving the problem with fixed complexity for a fixed dimension [17].

C. Stochastic Error with Imprecisely Known Correlations Only

For the case regarding known distributions with imprecisely known correlations, it has been shown in [18] that we can get a family of bounding ellipsoids in joint space in the form of

$$\mathbf{B}(\kappa) = \begin{bmatrix}\frac{1}{\eta(\kappa)-\kappa}\mathbf{C}_{xx} & \mathbf{0}\\ \mathbf{0} & \frac{1}{\eta(\kappa)+\kappa}\mathbf{C}_{yy}\end{bmatrix}$$

for $r_{max} \in (-1, 1)\backslash\{0\}$, $|\kappa| < 0.5$, and

$$\eta(\kappa) = \frac{1 - \sqrt{r^2_{max} + \kappa^2(1 - r^2_{max})^2}}{1 - r^2_{max}}$$

that properly bound

$$\mathrm{Cov}\left\{\begin{bmatrix}\underline{\boldsymbol{x}}\\ \underline{\boldsymbol{y}}\end{bmatrix}\right\} = \begin{bmatrix}\mathbf{C}_{xx} & \mathbf{C}_{xy}(\underline{r})\\ \mathbf{C}_{yx}(\underline{r}) & \mathbf{C}_{yy}\end{bmatrix} \leq \mathbf{B}(\kappa)$$

for all $\underline{r} \in \mathcal{R}$ and $\kappa \in (-0.5, 0.5)$.

Using the above, we can state that

$$\begin{aligned}&\forall\underline{r} \in \mathcal{R}, \kappa \in (-0.5, 0.5) : \mathbf{C}_z(\underline{r}) \leq \mathbf{V}(\kappa)\\ &\quad= \frac{1}{\eta(\kappa) - \kappa}\mathbf{H}_x\mathbf{C}_{xx}\mathbf{H}_x^T + \frac{1}{\eta(\kappa) + \kappa}\mathbf{H}_y\mathbf{C}_{yy}\mathbf{H}_y^T \ .\end{aligned}$$

Thus, the Mahalanobis distance of $\mathbf{H}_x\underline{x}$ and $\mathbf{H}_y\underline{y}$ is always less or equal if we use any $\mathbf{V}(\kappa)$ instead of any $\mathbf{C}_z(\underline{r})$ as the covariance matrix. Most importantly, as the bounds also satisfy condition (3), it is possible to determine the minimal Mahalanobis distance for all $\mathbf{C}(\underline{r})$ by determining the maximum of the Mahalanobis distances when $\mathbf{V}(\kappa)$ is used instead. All in all, the test presented in [3] performs the desired validation by testing

$$\begin{aligned}k \geq \max_{\kappa\in(-0.5,0.5)}&\Bigg\{(\mathbf{H}_x\hat{\underline{x}} - \mathbf{H}_y\hat{\underline{y}})^T\\ &\left(\frac{\mathbf{H}_x\mathbf{C}_{xx}\mathbf{H}_x^T}{\eta(\kappa) - \kappa} + \frac{\mathbf{H}_y\mathbf{C}_{yy}\mathbf{H}_y^T}{\eta(\kappa) + \kappa}\right)^{-1}(\mathbf{H}_x\hat{\underline{x}} - \mathbf{H}_y\hat{\underline{y}})\Bigg\} \ ,\end{aligned}$$

which only requires a convex optimization.

D. Stochastic Error with Known Correlations and Set-membership Error

Before considering imprecisely known correlations in presence of set-membership uncertainties, we present a technique for testing in the case of known correlations. In this case, the covariance $\mathbf{C}_z$ is known exactly. The condition

$$\exists\underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \mathbf{H}_x\underline{x} - \mathbf{H}_y\underline{y} \in \mathcal{E}(\underline{0}, k\mathbf{C}_z)$$

presented in subsection II-D can be formulated differently using the Mahalanobis distance as

$$\exists\underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \ d^2_\mathbf{W}(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \leq k \ . \tag{9}$$

Essentially, we have to minimize a weighted ellipsoid distance. This can be done by minimizing and testing

$$\min_{\underline{x}\in\mathcal{X},\underline{y}\in\mathcal{Y}} d^2_\mathbf{W}(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \leq k$$

under the constraints

$$\begin{aligned}(\underline{x} - \hat{\underline{x}})^T\mathbf{X}^{-1}(\underline{x} - \hat{\underline{x}}) &\leq 1 \text{ and}\\ (\underline{y} - \hat{\underline{y}})^T\mathbf{Y}^{-1}(\underline{y} - \hat{\underline{y}}) &\leq 1 \ ,\end{aligned}$$

which ensure both $\underline{x}$ and $\underline{y}$ are within the respective ellipsoids. By expanding the terms and taking into consideration that $\hat{\underline{x}}$ and $\hat{\underline{y}}$ are constant vectors, we get the constraints

$$\underline{x}^T\mathbf{X}^{-1}\underline{x} - 2\hat{\underline{x}}^T\mathbf{X}^{-1}\underline{x} + \hat{\underline{x}}^T\mathbf{X}^{-1}\hat{\underline{x}} - 1 \leq 0 \text{ and}$$
$$\underline{y}^T\mathbf{Y}^{-1}\underline{y} - 2\hat{\underline{y}}^T\mathbf{Y}^{-1}\underline{y} + \hat{\underline{y}}^T\mathbf{Y}^{-1}\hat{\underline{y}} - 1 \leq 0$$

in the standard form for quadratically constrained quadratic programs [19].

As $\mathbf{W}$, $\mathbf{X}$, and $\mathbf{Y}$ are at least positive semi-definite, this is a convex optimization problem. There are several software packages solving these kinds of optimization problems available —refer to [19] for suggestions. Although the global optimum can be searched via convex optimization, the algorithm may safely terminate and accept the measurement if any pair $(\underline{x}, \underline{y}) \in \mathcal{X} \times \mathcal{Y}$ is found that satisfies

$$d^2_{\mathbf{W}}(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \leq k \ .$$

Remark 2. As an alternative to computing the weighted ellipsoid distance, we can compute the common euclidean ellipsoid distance and implement the weighting by transforming the constraints. First, we can also formulate the constraints in the transformed space as

$$(\mathbf{H}_x\underline{x} - \mathbf{H}_x\hat{\underline{x}})^T(\mathbf{H}_x\mathbf{X}\mathbf{H}_x^T)^{-1}(\mathbf{H}_x\underline{x} - \mathbf{H}_x\hat{\underline{x}}) \leq 1 \text{ and}$$
$$(\mathbf{H}_y\underline{y} - \mathbf{H}_y\hat{\underline{y}})^T(\mathbf{H}_y\mathbf{Y}\mathbf{H}_y^T)^{-1}(\mathbf{H}_y\underline{y} - \mathbf{H}_y\hat{\underline{y}}) \leq 1 \ .$$

Second, we implement the weighting in the constraints. If we denote the Cholesky decomposition of $\mathbf{C}_z$ as $\mathbf{L}^T\mathbf{L}$ we may optimize and verify

$$\min_{\underline{x}\in\mathcal{X},\underline{y}\in\mathcal{Y}} d^2_2(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \leq k$$

with the constraints

$$(\mathbf{L}\mathbf{H}_x\underline{x} - \mathbf{H}_x\hat{\underline{x}})^T(\mathbf{H}_x\mathbf{X}\mathbf{H}_x^T)^{-1}(\mathbf{L}\mathbf{H}_x\underline{x} - \mathbf{H}_x\hat{\underline{x}}) - 1 \leq 0 \text{ and}$$
$$(\mathbf{L}\mathbf{H}_y\underline{y} - \mathbf{H}_y\hat{\underline{y}})^T(\mathbf{H}_y\mathbf{Y}\mathbf{H}_y^T)^{-1}(\mathbf{L}\mathbf{H}_y\underline{y} - \mathbf{H}_y\hat{\underline{y}}) - 1 \leq 0$$

instead.

E. Stochastic Error with Imprecisely Known Correlations and Set-membership Error

Having dealt with known correlations in the previous section, we now use the above insights for a condition to perform the validation in the case of imprecisely known correlations. In this case—similar to condition (9)—condition (7) becomes

$$\exists\underline{x} \in \mathcal{X}, \underline{y} \in \mathcal{Y} : \max_{\kappa\in(-0.5,0.5)} d^2_{\mathbf{W}(\kappa)}(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \leq k \quad (10)$$

with $\mathbf{W}(\kappa) = \mathbf{V}(\kappa)^{-1}$. We can also write this as the optimization problem

$$\min_{\underline{x}\in\mathcal{X},\underline{y}\in\mathcal{Y}} \left\{ \max_{\kappa\in(-0.5,0.5)} d^2_{\mathbf{W}(\kappa)}(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \right\} \leq k \ . \quad (11)$$

The algorithm for performing the above test was implemented in MATLAB and is shown in a MATLAB-like format in algorithm 1. Compared with the algorithms in the previous sections, this is an expensive one to solve numerically: for every function evaluation of $f(\underline{x}, \underline{y})$ necessary for solving the outer optimization loop, the inner convex optimization depending on κ has to be performed. Furthermore, despite the convexity of both optimization problems regarded separately, we cannot guarantee that the outer optimization is free of local minima in this form. Be aware that due to the inner optimization being performed, the outer optimization is in general no quadratic program. Nonetheless, our implementation that uses built-in MATLAB functions for solving the nonlinear program performed considerably well. As the optimization algorithm for the outer optimization, both MATLAB's interior point algorithm [20] and the Karush–Kuhn–Tucker equations [21] based solver were regarded.

Keeping in mind that the condition (10) is not aiming to minimize the type II error probability, a tempting way to perform an easier validation is by performing tests that may accept even more measurements. Designing and evaluating such algorithms may be a topic for prospective research.

Remark 3. For example swapping $\min$ and $\max$ would cause the test to become more conservative. While

$$\exists x\forall z : p(x, z) \rightarrow \forall z\exists x : p(x, z)$$

is logically valid for any predicate $p(x, z)$, the statement

$$\forall z\exists x : p(x, z) \rightarrow \exists x\forall z : p(x, z)$$

is only valid for certain predicates. In the above case, you can imagine that

$$\min_{\underline{x}\in\mathcal{X},\underline{y}\in\mathcal{Y}} d^2_{\mathbf{W}(\kappa)}(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \leq k \quad (12)$$

may be satisfied for any $\kappa \in (-0.5, 0.5)$, thus satisfying

$$\max_{\kappa\in(-0.5,0.5)} \left\{ \min_{\underline{x}\in\mathcal{X},\underline{y}\in\mathcal{Y}} d^2_{\mathbf{W}(\kappa)}(\mathbf{H}_x\underline{x}, \mathbf{H}_y\underline{y}) \right\} \leq k \ .$$

However, if different κ require different pairs $(\underline{x}, \underline{y}) \in \mathcal{X} \times \mathcal{Y}$ to fulfill the condition (12), then condition (11) is not satisfied.

IV. Example and Experiments

Now, let us look at an example for the above described data validation. We know from condition (8) in subsection II-E that the above test should consider a measurement $\mathcal{Y}$ valid iff

$$\mathbf{H}_y\mathcal{Y} \cap \left(\mathbf{H}_x\mathcal{X} \oplus \bigcup_{\underline{r}\in\mathcal{R}} \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r}))\right) \neq \emptyset \ . \quad (13)$$

In our example, we considered $\mathcal{X}$ and $\mathbf{C}_{xx}$ to be fixed. For the sake of being able to efficiently visualize the results, we considered $\mathbf{C}_{yy}$ to be fixed as well. k was set to 1 and the imprecisely known correlations were symmetrically bounded by $r_{max} = 0.7$. The other fixed parameters were chosen as

$$\hat{\underline{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$
$$\mathbf{X} = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} ,$$
$$\mathbf{H}_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \qquad \mathbf{H}_y = \begin{bmatrix} 1 & 0 \\ 0.1 & 2 \end{bmatrix} ,$$
$$\mathbf{C}_{xx} = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{bmatrix} \text{ and } \qquad \mathbf{C}_{yy} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix} .$$

As no optimization specific parameters were passed, MATLAB set these automatically. For example, for the Karush–Kuhn–Tucker based optimization, the limits for both the iteration and

```
function isValid=valTest($\hat{\underline{x}}$,$\mathbf{C}_{xx}$,$\mathbf{X}$,$\hat{\underline{y}}$,$\mathbf{C}_{yy}$,$\mathbf{Y}$,$\mathbf{H}_x$,$\mathbf{H}_y$,$r_{max}$,optimAlgo,k)
% Use 'active-set' as optimAlgo for Karush-Kuhn-Tucker equations based
% optimization or 'interior-point' for interior point based optimization
    % Use ellipsoid centers as starting points for the optimization
    v0=[$\hat{\underline{x}}$;$\hat{\underline{y}}$]; % As vector because they may not be of equal lengths
    options=optimset('Algorithm',optimAlgo);
    % Perform outer optimization
    [~,minDist]=fmincon(@objFun,v0,[],[],[],[],[],[],@conFun,options);
    isValid=minDist≤k;
    function d=objFun($\underline{v}$) % Inner optimization performed in objective
        $\underline{x}$=$\underline{v}$(1:length($\hat{\underline{x}}$)); $\underline{y}$=$\underline{v}$(length($\hat{\underline{x}}$)+1:end);
        $\eta$=@($\kappa$) $\frac{1-\sqrt{r_{max}^2+\kappa^2(1-r_{max}^2)^2}}{1-r_{max}^2}$; % Declare $\eta$ as a lambda function
        [~,d]=fminbnd(@($\kappa$) $-(\mathbf{H}_x\underline{x}-\mathbf{H}_y\underline{y})^T$ ... % Use negative to find max
            $\cdot(\frac{1}{\eta(\kappa)-\kappa}\mathbf{H}_x\mathbf{C}_{xx}\mathbf{H}_x^T+\frac{1}{\eta(\kappa)+\kappa}\mathbf{H}_y\mathbf{C}_{yy}\mathbf{H}_y^T)^{-1}(\mathbf{H}_x\underline{x}-\mathbf{H}_y\underline{y}), -.5, .5)$;
        d=-d;
    end
    function [c,ceq]=conFun($\underline{v}$) % Set constraint function
        $\underline{x}$=$\underline{v}$(1:length($\hat{\underline{x}}$)); $\underline{y}$=$\underline{v}$(length($\hat{\underline{x}}$)+1:end);
        c=[$\underline{x}^T\mathbf{X}^{-1}\underline{x}-2\hat{\underline{x}}^T\mathbf{X}^{-1}\underline{x}+\hat{\underline{x}}^T\mathbf{X}^{-1}\hat{\underline{x}}-1$; ...
           $\underline{y}^T\mathbf{Y}^{-1}\underline{y}-2\hat{\underline{y}}^T\mathbf{Y}^{-1}\underline{y}+\hat{\underline{y}}^T\mathbf{Y}^{-1}\hat{\underline{y}}-1$];
        ceq=[];
    end
end
```

Algorithm 1. MATLAB-like code for performing the validation.

function evaluations were automatically set to 400. In the case of the interior point based optimization, the iteration limit and function evaluation limits were 100 and 3000, respectively.

As an example, six measurements and their corresponding classification results are drawn in Fig. 7(a). For better visual judgment an approximation of

$$\mathbf{H}_x\mathcal{X} \oplus \bigcup_{\underline{r}\in\mathcal{R}} \mathcal{E}(\underline{0}, k\mathbf{C}_z(\underline{r}))$$

is shown in Fig. 7(b).

In order to assess the impact of the numerical and optimization related issues on the classification, we pseudorandomly generated a thousand measurements and performed the validation separately for both optimization algorithms. For the same measurements, we performed a test based on numerically approximating condition (13) to compare with. The numeric approximation was implemented by building the convex hull of points on the edges of several covariance ellipsoids, taking the Minkowski sum with $\mathbf{H}_x\mathcal{X}$, and then testing for an intersection with $\mathbf{H}_y\mathcal{Y}$. Whereas this may be an option for lower dimensions, it quickly gets unfeasible for higher dimensions. Using the Karush–Kuhn–Tucker based optimization, the classification of 992 out of 1000 measurements agreed with the results of the approximation of condition (13). When we used the interior point based implementation, the tests agreed 995 out of 1000 times. There were measurements that the approximation of condition (13) considered valid but the optimization based implementations did not—and also the other way around. Four out of the five times a disagreement occurred with the interior point based algorithm, the Karush–Kuhn–Tucker based implementation yielded the same result. Thus, inaccuracies in the approximation of condition (13) should also be considered as a source of error. Nonetheless, as the optimization based classifies did not always agree and differences in reported Mahalanobis distances were observed, we believe that local minima do in fact cause misclassifications for the above presented algorithm, regardless of the optimization algorithm employed. However, the misclassification rates were low and could be further reduced by using more robust optimization. While these results are generally suggesting that the algorithm works reasonably well in lower dimensions, a more rigorous evaluation also considering higher dimensions may be in order for further assessment of the algorithm's performance.

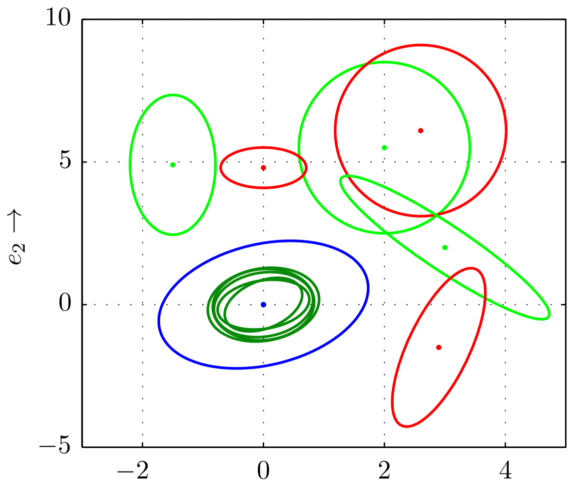
The source code for the optimization and an example can be viewed, run, and downloaded at http://www.cloudrunner.eu/algorithm/123. The example is similar to Fig. 7(b)—six measurements are generated pseudorandomly, with visual aids displayed for comparing to an approximation of condition (13).

V. Conclusion and Outlook

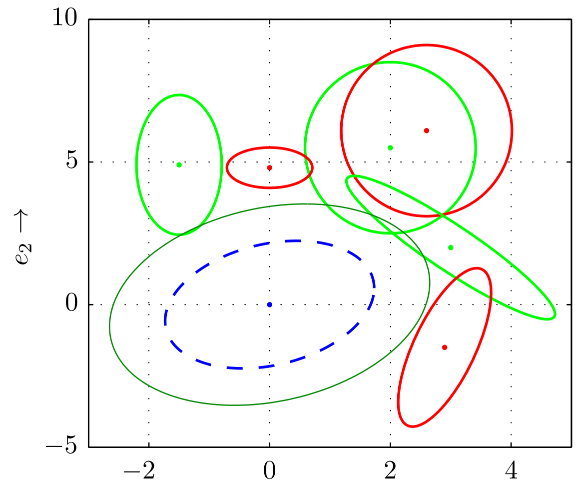
Data validation can be done in easy and efficient ways as long as the distributions and correlations of the errors are fully known. While previous work has already been concerned with relaxing the requirement for fully known correlations [3], we have shown the challenges of further regarding set-membership uncertainties.

Whereas performing data validation for purely set-membership uncertainties can be reduced to the common intersection test, additionally taking stochastic uncertainties into consideration requires more complex problems to be solved. For the frequently assumed case of known correlations, we have presented a test that performs the validation via convex optimization and for the case of imprecisely known correlations, we gave an example for an expensive algorithm.

Although further complexity is added, the tests allow for validation while modeling the perturbations with as much knowledge as available and without any need for further assumptions. Unsurprisingly, regarding a combined stochastic and set-membership error as being purely Gaussian simplifies the validation, but it may impede the quality of the actual validation performed.



(a) Some possible $1\mathbf{C}$-bound ellipsoids drawn centered in dark green with measurements and their validity in the corresponding colors.



(b) Minkowski sum of union of some possible $1\mathbf{C}$-bound ellipsoids and $\mathbf{H}_x\mathcal{X}$ drawn in dark green.

Figure 7. Classification results for an example. $\mathbf{H}_x\mathcal{X}$ in blue, measurements classified as valid in light green and measurements classified as invalid in red.

As the benefits of a better model for the uncertainties may still persist when the computationally expensive tests are approximated, future work will be concerned with fast approximations for the presented concepts. Quick tests that may only cause additional type II errors but no type I errors (as hinted at in remark 3) would still allow us to state an upper bound for the type I error probability. Instead of approximating the tests, they could also be sped up by criteria that allow for efficient classification in many cases, avoiding the expensive tests except for cases in which they are inevitable.

Opposite to techniques that potentially sacrifice probability of correctly classifying invalid measurements for easier calculation, tighter bounds as explained in remark 1 could be explored further. This can lead to a reduction of the type II error probability while still satisfying the predefined lower bound for the correct classification of valid measurements.

References

[1] Y. Bar-Shalom and T. E. Fortmann, *Tracking and Data Association*. Academic Press, 1988.

[2] G. Casella and R. L. Berger, *Statistical Inference*, 2nd ed. Cengage Learning, 2001.

[3] U. D. Hanebeck and J. Horn, "Data Validation in the Presence of Imprecisely Known Correlations," in *Proceedings of the 2003 European Control Conference 2003 (ECC 2003)*, Cambridge, United Kingdom, Sep. 2003.

[4] R. L. Berger, "Likelihood Ratio Tests and Intersection-Union Tests," *Advances in Statistical Decision Theory and Applications*, pp. 225–237, 1997.

[5] A. C. Bovik, *Handbook of Image and Video Processing*, 2nd ed. Academic Press, 2005.

[6] B. Noack, V. Klumpp, D. Brunn, and U. D. Hanebeck, "Nonlinear Bayesian Estimation with Convex Sets of Probability Densities," in *Proceedings of the 11th International Conference on Information Fusion (Fusion 2008)*, Cologne, Germany, Jul. 2008.

[7] B. Noack, V. Klumpp, and U. D. Hanebeck, "State Estimation with Sets of Densities considering Stochastic and Systematic Errors," in *Proceedings of the 12th International Conference on Information Fusion (Fusion 2009)*, Seattle, Washington, USA, Jul. 2009.

[8] A. Benavoli, M. Zaffalon, and E. Miranda, "Robust Filtering Through Coherent Lower Previsions," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1567–1581, July 2011.

[9] B. Noack, F. Pfaff, and U. D. Hanebeck, "Combined Stochastic and Set-membership Information Filtering in Multisensor Systems," in *Proceedings of the 15th International Conference on Information Fusion (Fusion 2012)*, Singapore, Jul. 2012.

[10] ——, "Optimal Kalman Gains for Combined Stochastic and Set-Membership State Estimation," in *Proceedings of the 51st IEEE Conference on Decision and Control (CDC 2012)*, Maui, Hawaii, USA, Dec. 2012.

[11] Y. Bar-Shalom, X.-R. Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*. John Wiley & Sons, 2001.

[12] F. L. Chernousko, *State Estimation for Dynamic Systems*. CRC Press, 1994.

[13] A. Kurzhanski and I. Vályi, *Ellipsoidal Calculus for Estimation and Control*. Birkhäuser Boston, 1996.

[14] F. C. Schweppe, *Uncertain Dynamic Systems*. Prentice-Hall, 1973.

[15] T. H. Kerr, "A Two Ellipsoid Overlap Test for Real Time Failure Detection and Isolation by Confidence Regions," in *Proceedings of the 13th IEEE Conference on Decision and Control including the 13th Symposium on Adaptive Processes (CDC 1974)*, Nov. 1974.

[16] S. Alfano, "Determining If Two Solid Ellipsoids Intersect," *Journal of Guidance, Control, and Dynamics*, vol. 26, pp. 106–110, 2003.

[17] I. Gilitschenski and U. D. Hanebeck, "A Robust Computational Test for Overlap of Two Arbitrary-dimensional Ellipsoids in Fault-Detection of Kalman Filters," in *Proceedings of the 15th International Conference on Information Fusion (Fusion 2012)*, Singapore, Jul. 2012.

[18] U. D. Hanebeck, K. Briechle, and J. Horn, "A Tight Bound for the Joint Covariance of Two Random Vectors with Unknown but Constrained Cross–Correlation," in *Proceedings of the 2001 IEEE International Conference on Multisensor Fusion and Integration for Intelligent Systems (MFI 2001)*, Baden–Baden, Germany, Aug. 2001.

[19] S. P. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[20] R. H. Byrd, J. C. Gilbert, and J. Nocedal, "A Trust Region Method Based on Interior Point Techniques for Nonlinear Programming," INRIA Rocquencourt, Rapport de recherche 2896, 1996.

[21] H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," in *Proceedings of 2nd Berkeley Symposium*, 1951.

[22] A. A. Kurzhanskiy and P. Varaiya, "Ellipsoidal Toolbox," EECS Department, University of California, Berkeley, Tech. Rep. UCB/EECS-2006-46, May 2006.