A Robust Computational Test for Overlap of Two Arbitrary-dimensional Ellipsoids in Fault-Detection of Kalman Filters

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***Abstract*—On-line fault-detection in uncertain measurement and estimation systems is of particular interest in many applications. In certain systems based on the Kalman filter, this test can be performed by checking whether hyperellipsoids overlap. This test can be applied to detecting failure in the system itself or in the sensors used to determine the system state. To facilitate the practical application of such tests, we describe a simple condition for overlap of two ellipsoids and propose an efficient algorithmic implementation for testing this condition. There are applications in many other areas, such as collision avoidance or computer graphics. Our proposal makes use of Leverriere's algorithm and Sturm's theorem, a result of algebraic geometry. Thus, no approximative methods, such as root finding or minimization are needed. Furthermore, the complexity of the algorithm is fixed for a fixed problem dimension.**

I. Introduction

This paper is concerned with fault-detection in digital control systems. Wrong behavior of systems with noisy measurements and a system model incorporating uncertainty, can lead to an infeasible system state, misleading information about the true system state, or wrong control decisions. In this paper, we are concerned with fault-detection in such systems. The development of efficient algorithms solving this problem makes on-line and real-time applications possible.

System faults can be understood in two possible ways. First, a fault can happen in the system itself. This happens when system components fail or do not behave in an expected way. In this situation the model loses its accuracy and validity in describing the real behavior of the system. Second, a fault can happen in the sensor or measurement system. This leads to a wrong estimate of the actual system state and possibly wrong control decisions. Thus, it is of interest to detect errors, which occur at unknown time, having an unknown intensity.

Our investigation is done in the setting of a linear system with Gaussian disturbances and a noisy linear measurement equation. In this situation, one of the possible approaches to fault-detection is a two ellipsoid intersection test, which reaches back to the seventies and was initially proposed by Kerr[12]. Detecting a fault is performed by comparing the true system evolution with an a priori estimate. In this setting, a Kalman filter is used to calculate an optimal estimate of the true system state. The a priori estimate is obtained without incorporation of the observations. Ellipsoidal confidence regions are placed around both estimates. Our test is based on checking whether both confidence regions intersect. This approach is conservative in recognizing possible system or measurement fault.

Especially in time-critical systems, an efficient and robust algorithm for checking ellipsoid overlap is important, as the efficiency of our failure-detection relies on the efficiency of the ellipsoid overlapping test. This test is usually performed by minimizing a specific function or by root finding. In the case of root finding, checking the ellipsoid intersection is easy for low problem dimensions. Solutions to polynomial equations can be given directly for polynomials of degrees lower than five. The Abel-Ruffini theorem states, that a general algebraic solution to this problem does not exist. For a direct test of overlap of higher dimensional ellipsoids, a different approach has to be taken.

A. Main Contribution

In this paper, we propose a new test for ellipsoidal overlap, which works for arbitrary high dimensions and does not make use of approximate methods. The execution time is fixed for a given system with fixed problem dimension. We formulate the question of ellipsoid overlapping as a convex optimization problem. This formulation is based on the ideas presented in [8]. A similar formulation was originally made in [9], [10], [17] for testing the intersection of two ellipsoids and for testing the intersection of an ellipsoid with a strip.

We interpret this problem formulation in a way that makes searching for the actual minimum unnecessary. It is sufficient to count roots of the convex function. We show that this is equivalent to counting distinct roots of a certain polynomial. Sturm's theorem is used, as it offers a way to do this without approximate methods.

Ellipsoids are often of special interest, because they offer a good approximation of convex objects. Thus, testing for overlap has applicability to many other areas, such as physics, computer graphics (e.g., game development and computer aided design), or collision avoidance.

The proposed intersection test is applied to a fault-detection scenario in the system described above. This application is of particular interest in the field of embedded or real-time systems, because limited ressources and a constrained computational time present an important challenge.

B. Related Work

1) Related work on fault-detection: Fault-detection is up to now an active area of research [7]. The approach discussed here is originally based on [12], [13], [14]. This work was extended algorithmically in [20], [21]. A slightly different approach to fault-detection is based on a χ^2-test, which was presented in [4].

2) Related work on ellipsoid overlap testing: In [15], an approximate method is presented for testing ellipsoid intersection. An algebraic condition similar to ours was developed in [19] with restriction to three dimensions. It makes use of root-counting as well, while taking a slightly different mathematical approach. We believe this condition also to be generalizable to the n-dimensional case. Testing ellipsoid overlap by observing eigenvalue behavior is proposed in [1].

C. Overview

In the next Section, we present some preliminary results. This are Sturm's theorem for polynomial root counting and Leverierre algorithm for computing the resolvent of a matrix. In Section III, we consider a linear system and formulate the failure condition. This condition is used to motivate the ellipsoid overlapping test. Section IV contains the derivation of a simple condition for ellipsoid intersection. A robust algorithmic implementation of the test is proposed in Section V. Finally, we conclude this paper in Section VI and give an outlook of further research.

II. Preliminaries

A. Sturm's Theorem

Sturm's theorem is a result from algebraic geometry, which enables counting real roots of polynomials in a given interval without explicitly computing them. The basic idea is generating a special sequence of polynomials, the so-called Sturm sequence. The first two elements of this sequence are the polynomial of interest and its derivative. The remaining elements are generated using the Euclidean algorithm. Root counting is implemented by evaluating the whole sequence of polynomials at the interval borders and comparing the number of sign changes in the computed value sequences.

Counting all real roots in a given interval using Sturm's theorem is only possible for square-free polynomials, i.e., polynomials without repeated roots. The result can be generalized to polynomials with repeated roots, when one is only interested in distinct roots. For this general case Sturm's theorem still holds in the sense, that roots are counted without multiplicity. First we define the Sturm sequences (often also called Sturm chains) as described above.

Definition 1: Let $p(\lambda)$ be a square-free polynomial and let $\mathrm{rem}(p, q)$ denote the remainder from a polynomial division of p and q. The sequence of polynomials

$$\begin{aligned} p_0(\lambda) &:= p(\lambda) \ , \\ p_1(\lambda) &:= p'(\lambda) \ , \\ p_2(\lambda) &:= -\mathrm{rem}(p_0, p_1) \ , \\ &\vdots \\ p_n(\lambda) &:= -\mathrm{rem}(p_{n-2}, p_{n-1}) \end{aligned} \tag{1}$$

is called the Sturm sequence of $p(\lambda)$.

Theorem 1 (Sturm): Let $a, b \in \mathbb{R}$ with $a < b$ and let $p \in \mathbb{R}[\lambda]$ be square-free. The number of real roots of $p(\lambda)$ in (a, b) is given by $\sigma_p(a) - \sigma_p(b)$, where $\sigma_p(a)$ is the number of sign changes of the Sturm sequence of $p(\lambda)$ evaluated at a.

For an efficient implementation, it is not necessary to keep the whole Sturm sequence in memory. It is sufficient to evaluate each polynomial in the sequence immediately after its computation. See [3] and [5] for proofs and a further discussion of Sturm's theorem.

B. Leverriere Algorithm

The resolvent of a matrix plays an important role in control engineering e.g., in the solution of state-space equations by Laplace transformation [6]. For a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, its resolvent is defined as

$$\mathbf{R}(\lambda) = (\lambda\mathbf{I} - \mathbf{A})^{-1} \ ,$$

where $\mathbf{I}$ is the identity matrix and $\lambda \in \mathbb{R}$. This can be decomposed into

$$(\lambda\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathrm{adj}(\lambda\mathbf{I} - \mathbf{A})}{\det(\lambda\mathbf{I} - \mathbf{A})} \ . \tag{2}$$

Thus, the resolvent is a rational matrix, i.e., each entry is a ratio of two polynomial functions. Leverriere algorithm is used to compute the determinant and the adjugate of $(\lambda\mathbf{I} - \mathbf{A})$ simultaneously. At each step of the algorithm, one coefficient of the determinant and one coefficient matrix of the adjugate is calculated.

The determinant of $\lambda\mathbf{I} - \mathbf{A}$ is a polynomial of degree n and the respective adjugate is a polynomial matrix, where each entry has at most degree $n - 1$. They can be written as

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 \ , \\ \mathrm{adj}(\lambda\mathbf{I} - \mathbf{A}) &= \mathbf{T}_{n-1}\lambda^{n-1} + \mathbf{T}_{n-2}\lambda^{n-2} + \cdots + \mathbf{T}_0 \ , \end{aligned}$$

where $a_i \in \mathbb{R}$ and $\mathbf{T}_i \in \mathbb{R}^{n\times n}$.

One can transform (2) into

$$\det(\lambda\mathbf{I} - \mathbf{A})\ \mathbf{I} = (\lambda\mathbf{I} - \mathbf{A})\ \mathrm{adj}(\lambda\mathbf{I} - \mathbf{A}) \ ,$$

so that

$$\begin{aligned} &\mathbf{I}\lambda^n + a_{n-1}\mathbf{I}\lambda^{n-1} + \cdots + a_1\mathbf{I}\lambda + a_0\mathbf{I} = \\ &\qquad \mathbf{T}_{n-1}\lambda^n + (\mathbf{T}_{n-2} - \mathbf{T}_{n-1}\mathbf{A})\lambda^{n-1} + \cdots + \mathbf{T}_0\mathbf{A} \ . \end{aligned}$$

Equating the coefficients yields formulas for $\mathbf{T}_i$ and a_i. The highest-order coefficient a_{n-1} and the coefficient matrix $\mathbf{T}_{n-1}$ can be obtained directly by

$$\mathbf{T}_{n-1} = \mathbf{I} \ , \qquad a_{n-1} = -\mathrm{tr}(\mathbf{A}) \ .$$

For the remaining coefficients, a backward iteration is used, where

$$\mathbf{T}_i = \mathbf{T}_{i+1}\mathbf{A} + a_{i+1}\mathbf{I} \ , \qquad a_i = -\frac{1}{n-i}\mathrm{tr}(\mathbf{T}_i\mathbf{A}) \ ,$$

and $0 \leq i < n-1$.

These formulas can be used for a straight-forward implementation. See [6, p. 246] for a further discussion of Leverriere algorithm.

III. Fault-Detection in Kalman Filters

Our fault-detection system is based on a two confidence region approach [14]. Making two estimates of the true system state is our strategy in this approach. The first one is the classical Kalman filter estimate after the measurement and the second one is an a priori estimate not incorporating any knowledge obtained from measurements and observations. This can be seen as comparing a system model in a closed-loop and an open-loop scenario. A confidence region is formed around both estimates. These regions are represented through ellipsoids, because they are easy to derive from the covariance matrices and the means of the respective estimates. The dimension of the ellipsoids corresponds to the dimension of the state space.

In our fault-detection test, these confidence regions are used for testing the hypotheses

"H_0: The system is in a feasible state."

"H_1: The system is in an infeasible state."

against each other. The existence of an intersection of the confidence regions serves as the test statistic. H_0 is accepted if both confidence regions share at least one common point. Thus, this test has a conservative behavior and prefers to assume that the system is in a feasible state. It should be applied to problems, where system failure leads to a very high deviation of the measurement from the a priori estimate.

Our considered system model is given by

$$\underline{x}(t+1) = \mathbf{\Phi}(t+1,t)\underline{x}(t) + \underline{\omega}(t+1) \ ,$$

with state vector $\underline{x}(t) \in \mathbb{R}^n$, state transition matrix $\mathbf{\Phi}(t+1,t) \in \mathbb{R}^{n\times n}$, and a Gaussian uncertainty $\underline{\omega}(t) \sim \mathcal{N}(\underline{0}, \mathbf{Q}(t))$. The measurement equation of this system is

$$\underline{z}(t) = \mathbf{H}(t)\underline{x}(t) + \underline{v}(t) \ ,$$

where the observed value is $\underline{z}(t) \in \mathbb{R}^n$, the output matrix is $\mathbf{H}(t) \in \mathbb{R}^{n\times n}$, and the uncertainty of the measurement is $\underline{v}(t) \sim \mathcal{N}(\underline{0}, \mathbf{R}(t))$.

In the following, $\underline{x}_1(t)$ shall denote the Kalman filter state estimation and $\underline{x}_2(t)$ the a priori state prediction. The state estimates are given by

$$\begin{aligned}\underline{x}_1(t+1) &= \mathbf{\Phi}(t+1,t)\underline{x}_1(t) + \\ &\quad \mathbf{K}(t+1)[\underline{z}(t+1) - \mathbf{H}(t+1)\mathbf{\Phi}(t+1,t)\underline{x}_1(t)] \ , \\ \underline{x}_1(0) &= \underline{x}_0 + \underline{\omega}(0) \ ,\end{aligned}$$

and

$$\begin{aligned}\underline{x}_2(t+1) &= \mathbf{\Phi}(t+1,t)\underline{x}_2(t) \ , \\ \underline{x}_2(0) &= \underline{x}_0 + \underline{\omega}(0) \ .\end{aligned}$$

The corresponding state covariances are

$$\begin{aligned}\mathbf{C}_1(t+1) &= \mathbf{\Phi}(t+1,t)\mathbf{W}(t)\mathbf{\Phi}^T(t+1,t) + \mathbf{Q}(t+1) \ , \\ \mathbf{C}_2(t+1) &= \mathbf{\Phi}(t+1,t)\mathbf{C}_2(t)\mathbf{\Phi}^T(t+1,t) + \mathbf{Q}(t+1) \ ,\end{aligned}$$

where $\mathbf{W}(t) := [\mathbf{I} - \mathbf{K}(t)\mathbf{H}(t)]\mathbf{C}_1(t)$ and $\mathbf{K}(t)$ is the Kalman gain

$$\mathbf{K}(t) = \mathbf{C}_1(t)\mathbf{H}^T(t)[\mathbf{H}(t)\mathbf{C}_1(t)\mathbf{H}^T(t) + \mathbf{R}(t)]^{-1}.$$

Two covariance bounds need to be found circumscribing our model estimates for two given probability levels $p_1, \ p_2$. This is equivalent to finding $M_1, M_2 \in \mathbb{R}$, so that

$$\mathbf{P}_1\left((\underline{x}(t) - \underline{x}_1(t))^T\mathbf{C}_1(t)^{-1}(\underline{x}(t) - \underline{x}_1(t)) \leq M_1\right) = p_1 \ ,$$

$$\mathbf{P}_2\left((\underline{x}(t) - \underline{x}_2(t))^T\mathbf{C}_2(t)^{-1}(\underline{x}(t) - \underline{x}_2(t)) \leq M_2\right) = p_2 \ ,$$

where $\mathbf{P}_i$ are the probability measures of our respective models with $\underline{x}(t) \sim \mathcal{N}(\underline{0}, \mathbf{C}_i(t))$. The values of $M_1, M_2 \in \mathbb{R}$ can be easily obtained with the help of the χ^2-distribution. For the construction of a test, we define the sets

$$\mathcal{E}_1 = \left\{\underline{x} \in \mathbb{R}^n | (\underline{x} - \underline{x}_1(t))^T(M_1\mathbf{C}_1(t))^{-1}(\underline{x} - \underline{x}_1(t)) < 1\right\} ,$$

$$\mathcal{E}_2 = \left\{\underline{x} \in \mathbb{R}^n | (\underline{x} - \underline{x}_2(t))^T(M_2\mathbf{C}_2(t))^{-1}(\underline{x} - \underline{x}_2(t)) < 1\right\} .$$

The test is performed by checking $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$ at each time step. As long as both ellipsoids overlap, H_0 is assumed to be true. This has a probabilistic interpretation. Denote X as the event that the true state is within the p_2 bound around $\underline{x}_2(t)$ and denote Y as the event, that the true state is within the p_1 bound around $\underline{x}_1(t)$. In the case, where our covariance bounds do not overlap, we have

$$\begin{aligned}\hat{\mathbf{P}}(X) &= \hat{\mathbf{P}}(X \cap Y) + \hat{\mathbf{P}}(C \cap \neg Y) \\ &= \hat{\mathbf{P}}(X \cap \neg Y) \\ &\leq \hat{\mathbf{P}}(\neg Y) \\ &= 1 - p_1 \ ,\end{aligned}$$

where $\hat{\mathbf{P}}$ is the conditional probability given that the confidence ellipsoids do not overlap.

IV. Ellipsoidal Calculus for Testing Ellipsoid Intersection

The development of a mathematical intersection condition for two ellipsoids is performed in two steps. First, all shared points of two ellipsoids are considered and an ellipsoid is circumscribed around them. Each of the circumscribing ellipsoids used here is a subset of the union of both ellipsoids.

Second, we try to minimize this circumscribed ellipsoid. In contrary to the strategy used in [11], our ellipsoids do not necessarily have the same center. If this minimization procedure leads to an infeasible result, it can be assumed that both ellipsoids do not intersect. Finally, this feasibility check

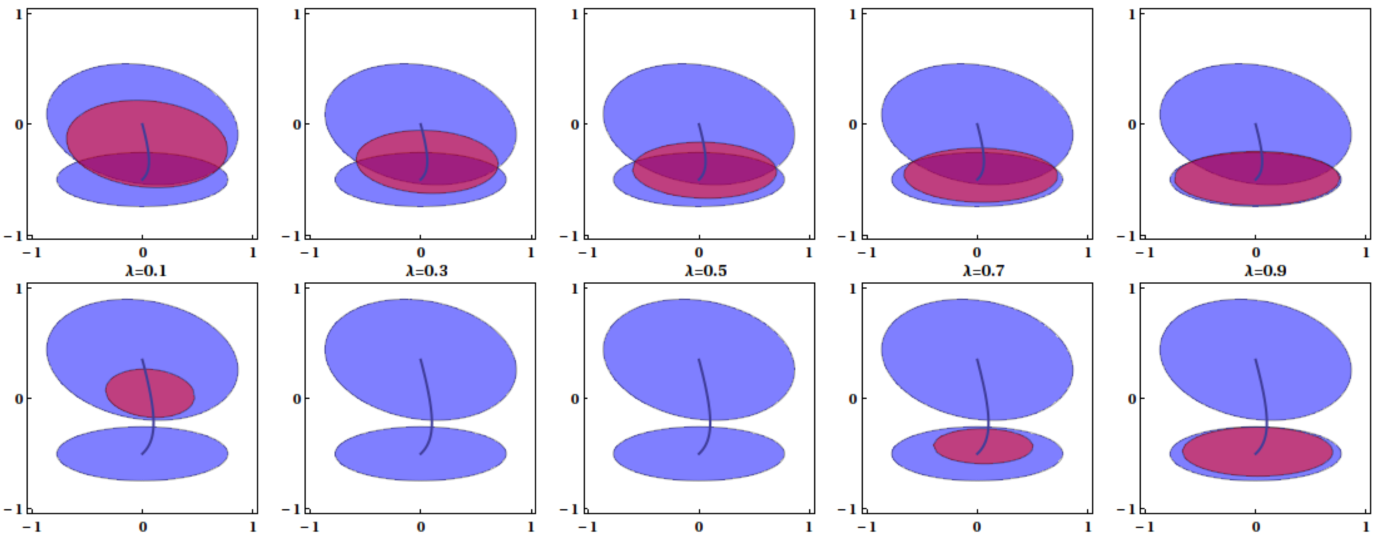

Figure 1. The ellipsoids described by proposition 1 are shown in red. They coincide with the respective original ellipsoids for $\lambda = 0$ or $\lambda = 1$. The red ellipsoids disappear for certain λ if both blue ellipsoids do not intersect. The curve connecting the centers of both blue ellipsoids is the path of m_λ

can be done through root counting or finding the minimum of a convex function and checking if its larger than zero.

The mathematical formulation of our intersection condition is based on an ellipsoid representation, which fits naturally into the representation used by covariance matrices in the normal distribution.

Notation 1: Let $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$ be symmetric positive definite and $\underline{\mu} \in \mathbb{R}^n$. Then the ellipsoid described by the set

$$\{\underline{x} \in \mathbb{R}^n : (\underline{x} - \underline{\mu})^T \mathbf{\Sigma} (\underline{x} - \underline{\mu}) \le 1\}$$

is denoted by $\mathcal{E}(\mathbf{\Sigma}, \underline{\mu})$.

For properly chosen $\underline{c}, \underline{d}, \mathbf{A}, \mathbf{B}$, our problem formulation in the previous section is equivalent to checking the intersection of the ellipsoids $\mathcal{E}_1 := \mathcal{E}(\mathbf{A}, \underline{c})$ and $\mathcal{E}_2 := \mathcal{E}(\mathbf{B}, \underline{d})$. The matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ are symmetric positive definite and $\underline{c}, \underline{d} \in \mathbb{R}^n$.

A. Condition for Overlap

Every point within the intersection $\underline{x} \in \mathcal{E}_1 \cap \mathcal{E}_2$ obviously satisfies

$$\lambda(\underline{x} - \underline{c})^T \mathbf{A} (\underline{x} - \underline{c}) + (1-\lambda)(\underline{x} - \underline{d})^T \mathbf{B} (\underline{x} - \underline{d}) \le 1 \quad (3)$$

for $\lambda \in [0, 1]$. Every point $\underline{x}$ satisfying this equation lies within the union of both ellipsoids. It is, however, not necessarily within the intersection. For $\lambda = 1$, this equation describes $\mathcal{E}_1$ and for $\lambda = 0$ it describes $\mathcal{E}_2$. The following proposition shows that the area described by (3) has ellipsoidal shape or is an empty set.

Proposition 1: Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be ellipsoids as defined above. The set of points satisfying (3) for $\lambda \in (0, 1)$ is either empty, or one single point, or an ellipsoid $\hat{\mathcal{E}}_\lambda := \mathcal{E}(\hat{\mathbf{E}}_\lambda^{-1}, \underline{m}_\lambda)$, where

$$\begin{aligned}
\hat{\mathbf{E}}_\lambda &= \mathbf{E}_\lambda / K(\lambda) , \\
\mathbf{E}_\lambda &= \lambda \mathbf{A} + (1-\lambda)\mathbf{B} , \\
\underline{m}_\lambda &= \mathbf{E}_\lambda^{-1}(\lambda \mathbf{A}\underline{c} + (1-\lambda)\mathbf{B}\underline{d}) , \\
K(\lambda) &= 1 - \lambda \underline{c}^T \mathbf{A} \underline{c} - (1-\lambda)\underline{d}^T \mathbf{B} \underline{d} + m_\lambda \mathbf{E}_\lambda m_\lambda .
\end{aligned}$$

Proof: By an algebraic transformation, (3) can be transformed into

$$(\underline{x} - \underline{m}_\lambda)^T \mathbf{E}_\lambda (\underline{x} - \underline{m}_\lambda) \le K(\lambda). \quad (4)$$

This is an ellipsoid for $K(\lambda) > 0$, because $\mathbf{E}_\lambda$ is positive definite and division by $K(\lambda)$ yields our ellipsoid formula. For $K(\lambda) = 0$, this set contains only $\underline{m}_\lambda$. For $K(\lambda) < 0$, division of (4) by $K(\lambda)$ results into an inequality, which is not satisfied by any $\underline{x} \in \mathbb{R}^n$. ■

Even if $\mathcal{E}_1$ and $\mathcal{E}_2$ do not overlap, the ellipsoid $\hat{\mathcal{E}}_\lambda$ described by this proposition exists for λ close to 0 and 1. It always satisfies

$$\mathcal{E}_1 \cap \mathcal{E}_2 \subseteq \hat{\mathcal{E}}_\lambda \subseteq \mathcal{E}_1 \cup \mathcal{E}_2 .$$

We focus our further attention on $K(\lambda)$, as it will be of particular importance for our overlap test.

Remark 1: Another known representation [16] of $K(\lambda)$ is

$$K(\lambda) = 1 - \lambda(1-\lambda)(\underline{d} - \underline{c})^T \mathbf{B} \mathbf{E}_\lambda^{-1} \mathbf{A} (\underline{d} - \underline{c}) ,$$

which is equivalent to

$$K(\lambda) = 1 - \underline{v}^T \left(\frac{1}{1-\lambda} \mathbf{B}^{-1} + \frac{1}{\lambda} \mathbf{A}^{-1} \right)^{-1} \underline{v} ,$$

where $\underline{v} = \underline{d} - \underline{c}$. It is, however, important to point out that $K(\lambda)$ is convex.

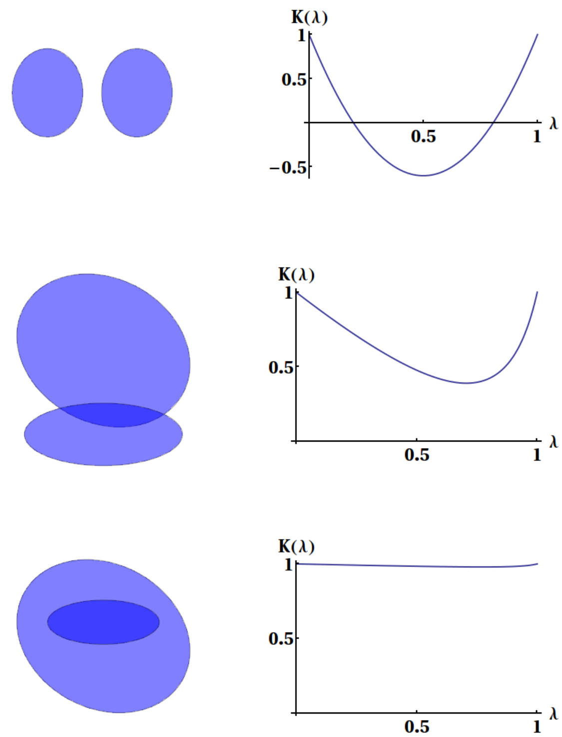

Figure 2. $K(\lambda)$ is convex and only becomes negative for some λ on $(0,1)$, if both ellipsoids do not overlap.

Proposition 2: The ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ share no common point if and only if $\lambda^* \in (0,1)$ exists with $K(\lambda^*) < 0$, where $K(\lambda)$ is defined as in proposition 1.

Proof: We assume the existence of $\lambda^* \in (0,1)$, so that $K(\lambda^*) < 0$. According to proposition 1, the existence of an intersection would imply the existence of an circumscribing ellipsoid with center $\underline{m}_{\lambda^*} \in \mathcal{E}_1 \cup \mathcal{E}_2$. From $K(\lambda^*) < 0$ it follows that no point satisfies (4) for our fixed λ^* and thus, neither a circumscribing ellipsoid nor common points exist.

Now, we assume that no intersection exists. In this case, we observe that $\underline{m}_\lambda$ is still a continuous function on $[0,1]$ leading from $\underline{m}_0 = \underline{d}$ to $\underline{m}_1 = \underline{c}$. From this continuity follows the existence of $\lambda^* \in (0,1)$, where $\underline{m}_{\lambda^*} \notin \mathcal{E}_1 \cup \mathcal{E}_2$. It follows that $\underline{x} = \underline{m}_{\lambda^*}$ must not satisfy (4). This can only be the case for $K(\lambda^*) < 0$. ∎

B. Example

We consider the covariance matrices

$$\mathbf{C}_1 = \begin{pmatrix} 0.75 & -0.08 \\ -0.08 & 0.3 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.06 \end{pmatrix}.$$

The ellipsoids $\mathcal{E}(\mathbf{C}_1^{-1}, (0,0)^T)$ and $\mathcal{E}(\mathbf{C}_2^{-1}, (0,-0.5)^T)$ intersect. The ellipsoid, which is bounding the intersection region never vanishes (shown in the first line of figure 1). If the position of the first ellipsoid is altered to $\mathcal{E}(\mathbf{C}_1^{-1}, (0,0.35)^T)$, then both ellipsoids do not intersect anymore and thus, the bounding ellipsoid disappears for certain λ (shown in the second line of figure 1).

V. Algorithmic Implementation of The Intersection Test

In this section, the preceding mathematical results will be used for developing an algorithm that tests the existence of an intersection of arbitrary n-dimensional ellipsoids. This algorithm checks the condition formulated in proposition 2.

Using the convexity of $K(\lambda)$, there are two possible strategies. First, it would be possible to find a minimum of $K(\lambda)$ using a convex optimization method. Finding the minimum of $K(\lambda)$ is, however, not necessary for the test. It is sufficient to check, whether $K(\lambda)$ has two distinct real roots on $(0,1)$. The existence of two real roots implies (due to convexity of) the existence of $\lambda^* \in (0,1)$ where $K(\lambda^*) < 0$.

Our proposed strategy is counting roots of the rational function $K(\lambda)$ on $(0,1)$, which is equivalent to counting roots of the polynomial $\det(\mathbf{E}_\lambda)K(\lambda)$, because $\det(\mathbf{E}_\lambda)$ has no roots on $(0,1)$. This reduces the basic problem to counting the roots of a polynomial instead of a rational function. The following transformation is made

$$\begin{aligned}\det(\mathbf{E}_\lambda) &= \det(\lambda\mathbf{A} + (1-\lambda)\mathbf{B}) \\ &= \det(\lambda(\mathbf{A}-\mathbf{B}) + \mathbf{B}) \\ &= \det(\lambda\mathbf{I} - (-)\mathbf{B}(\mathbf{A}-\mathbf{B})^{-1})\det(\mathbf{A}-\mathbf{B}) ,\end{aligned}$$

where $\mathbf{A}$ and $\mathbf{B}$ are defined in the same way as in proposition 1. This makes Leverriere algorithm applicable, which yields

$$\det(\lambda\mathbf{I} - (-\mathbf{B})(\mathbf{A}-\mathbf{B})^{-1}) , \quad \mathrm{adj}(\lambda\mathbf{I} - (-\mathbf{B})(\mathbf{A}-\mathbf{B})^{-1}) .$$

The complete overlap test is presented in figure 3. This approach works only if $\mathbf{A}-\mathbf{B}$ is invertible. Otherwise it has to be preceded by a modified column reduction method. For the ease of implementation, we used a method based on singular value decomposition (see [2]). This results in a non-direct method for testing ellipsoid overlap, because numerical approximation methods are used in the calculation of the singular value decomposition. A direct method can be obtained by using LU decomposition or modifications of some known column reduction methods (such as [18]).

The number of operations does not depend on the mutual position of both considered ellipsoids. Thus, the complexity of the given algorithm is solely dependent on the problem dimensionality n. The most expensive operation within the main loop is the matrix multiplication. The loop itself is executed n times. Using the straight forward matrix multiplication algorithm, the complexity of our whole algorithm is $\mathcal{O}(n^4)$. For the more general case, where $\mathbf{A}-\mathbf{B}$ is not invertible, our upper complexity bound scales up to $\mathcal{O}(n^5)$.

VI. Conclusion

We presented a new algorithm for checking ellipsoid intersection and applied it to on-line fault-detection in Kalman

Input: $\mathcal{E}_1 = \mathcal{E}(\mathbf{A}, \underline{c}),\ \mathcal{E}_2 = \mathcal{E}(\mathbf{B}, \underline{d})$
Output: $r = 2$ if $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$,
$r = 0$ or $r = 1$ otherwise
% Initialization
$\underline{m}_1 \leftarrow \mathbf{A}\underline{c} - \mathbf{B}\underline{d}$
$\underline{m}_2 \leftarrow \mathbf{B}\underline{d}$
$\mathbf{N} \leftarrow (\mathbf{A} - \mathbf{B})^{-1}$
$\mathbf{M} \leftarrow -\mathbf{B}\mathbf{N}$
$\mathbf{T}_{old} \leftarrow \mathbf{0}$
$a_n \leftarrow 1$
$i \leftarrow n$
% A slightly modified Leverierre algorithm.
while $i > 0$ **do**
  $\mathbf{T} \leftarrow (\mathbf{M}\mathbf{T}_{old}) + a_i\mathbf{I}$
  $P_{i-1} \leftarrow P_{i-1} + \underline{m}_2^T \mathbf{N}\mathbf{T}\underline{m}_2$
  $P_i \leftarrow P_i + 2\underline{m}_1^T \mathbf{N}\mathbf{T}\underline{m}_2$
  $P_{i+1} \leftarrow P_{i+1} + \underline{m}_1^T \mathbf{N}\mathbf{T}\underline{m}_1$
  $\mathbf{T}_{old} \leftarrow \mathbf{T}$
  $i \leftarrow i - 1$
  $a_i \leftarrow -\mathrm{tr}(\mathbf{M}\mathbf{T})/(n - i)$
end while
$S(\lambda) \leftarrow \det(\mathbf{E}_\lambda)(1 - \lambda\underline{c}^T\mathbf{A}\underline{c} - (1-\lambda)\underline{d}^T\mathbf{B}\underline{d})$
% $P(\lambda)$ is set to $det(\mathbf{E}_\lambda)K(\lambda)$
$P(\lambda) \leftarrow P(\lambda) + S(\lambda)$
$r \leftarrow$ count roots of $det(\mathbf{E}_\lambda)K(\lambda)$ on $(0, 1)$
return r

Figure 3. Algorithm for checking overlap of ellipsoids in the simplified case, where $\det(\mathbf{A} - \mathbf{B}) \neq 0$. In this algorithm, P_i are the coefficients of λ^i in $P(\lambda)$ and a_i are the coefficients of λ^i in $\det(\mathbf{E}_\lambda)$. Comment lines begin with a % sign.

filters. It is of particular interest for both, real-time control systems and embedded control systems, where simplicity of implementation is desirable. Our fault-detection test may also be applied in situations, where measurement or system errors result in extreme variable values (e.g., when there is an overflow or the memory is corrupted).

Future research can be done in three different areas. First, generalizing this fault-detection to other filters and control systems is of interest. This is particularly the case, where ellipsoids offer a reasonable description of uncertainty. Second, implementing the intersection test in other areas of application is also of interest. Third, improving the runtime of the algorithm itself is important to address problems in higher dimension.

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