Finite-horizon Dynamic Compensation of Markov Jump Linear Systems without Mode Observation

Maxim Dolgov, Gerhard Kurz, and Uwe D. Hanebeck

***Abstract*— In this paper, we address finite-horizon optimal control of Markov Jump Linear Systems with non-observed discrete-valued system state via dynamic output feedback. It has been shown that the optimal control law for this problem is intractable. For this reason, we assume a mode-independent control policy consisting of a linear time-variant estimator that reconstructs the continuous-valued state from noisy measurements, and a linear time-variant regulator that maps the state estimate to control inputs. To the best of our knowledge, this problem remained unsolved because even under the assumption of a linear control law, the separation between the estimator and the regulator does not hold. However, by minimizing an upper bound on the true costs induced by the control, we are able to design an iterative algorithm for computation of the controller parameters whose convergence is shown. The proposed algorithm is demonstrated by means of a simulation.**

	horizon	**feedback**	**time-**	**noise**
do Val et al. [1]	finite	state	variant	no
Vargas et al. [5]	finite	state	variant	yes
Vargas et al. [14]	finite	state	invariant	no
Vargas et al. [13]	finite	output	variant	yes
do Val et al. [15]	infinite	state	invariant	yes
Oliveira et al. [6]	infinite	state	invariant	yes
Dolgov et al. [16]	infinite	state	invariant	yes
Dolgov et al. [17]	infinite	output	invariant	yes
Fioravanti et al. [18][1]	infinite	noisy output	invariant	yes
Dolgov et al. [19]	infinite	noisy output	invariant	yes
this work	finite	noisy output	variant	yes
Pan et al. [20]	infinite	state	variant	yes
Campo et al. [21]	infinite	noisy output	variant	yes

TABLE I

OVERVIEW OF RELATED WORK.

I. INTRODUCTION

In many practical applications, the underlying nonlinear dynamical system model can often be simplified by defining a set of linear dynamical systems that are switched according to the state of a Markov chain with a finite state space. Such models constitute a special class of hybrid systems, namely the Markov Jump Linear Systems (MJLS). This modeling approach is especially suitable for describing systems with abrupt changes in their dynamics as it is, e.g., the case in economics [1], [2], networked control [3], [4], control of systems with component failures [5], [6], and many more [7].

Stochastic optimal control of MJLS has been of interest since their introduction in 1961 [8]. The most fundamental insight of this research was that, although the individual dynamical systems of an MJLS are linear, the optimal control policy is only linear if the discrete-valued state can be observed [9], [10]. Otherwise, the optimal policy is nonlinear. Furthermore, there is *dual effect*, which leads to intractability of the policy [11], [12]. For this reason, approximate but tractable control policies are of interest.

A popular class of approximate control policies for discrete-time MJLS with non-observed mode are based on the assumption of a linear control law. This approach was introduced in [1], where the authors considered finite-horizon optimal control via state feedback (we will refer to the continuous-valued state of the MJLS simply as the *state* and to the discrete-valued state as the *mode*). In this scenario, the control law is a time-variant linear regulator. The work [1] has been extended in [5] to MJLS with independent and identically distributed white Gaussian process noise. An extension thereof to static output-feedback control was presented in [13]. A finite-horizon time-invariant control of MJLS without process noise and mode observation via state feedback was addressed in [14], where the authors also evaluated several nonlinear optimization algorithms for computation of the regulator parameters.

The assumption of a linear control policy for MJLS was also applied to infinite-horizon control. In [15], the authors addressed state-feedback control with process noise for clustered mode observation, i.e., the modes are either observable or not, where a bound on the H_2 norm of the MJLS is minimized in order to obtain the regulator gain. If the cluster of the observed modes is empty, the scenario with no mode observation is recovered. A tighter H_2 bound was derived in [6]. In [18], the authors considered H_∞ dynamic compensation of MJLS and provided a solution in terms of an LMI for the case when the mode is observed. They also discussed the case of clustered mode observation. However, the proposed approach is either very conservative due to additional constraints or has to make restrictive assumptions on the form of the system matrices (no jumps in system matrices within each cluster, i.e., these matrices must be deterministic in case of no mode observation).

In our previous work, we considered optimal control of MJLS with process noise but no mode observation in [16], [17], [19]. In [16], we addressed state-feedback control and in [17] the static output-feedback scenarios, and presented iterative algorithms for the computation of the control laws.

The authors are with the Intelligent Sensor-Actuator-Systems Laboratory (ISAS), Institute for Anthropomatics and Robotics, Karlsruhe Institute of Technology (KIT), Germany. E-Mail: maxim.dolgov@kit.edu, gerhard.kurz@kit.edu, uwe.hanebeck@ieee.org

[1] Considers only a very restrictive scenario.

Our main contribution to infinite-horizon control of MJLS without mode observation was published in [19], where we addressed dynamic compensation for MJLS without mode observation and proposed an algorithm for the computation of a time-invariant controller that consists of a linear estimator and a linear regulator.

Finally, the authors of [20] and [21] take a different approach by applying principles of adaptive control to stochastic control of MJLS without mode observation. The work [20] considers state feedback, while noisy output-feedback is considered in [21]. In this framework, a controller with assumed mode observation is designed first. Then, a multiple model filter is used to compute a mode estimate at each time step. Different strategies for the actual controller are proposed, such as averaging and Maximum A Posteriori. An overview of the related work is given in Table I. In this table, the controller for noisy output feedback are dynamic.

In this paper, we consider the same scenario as in [22]. However, unlike [22], we do not assume that the controller has access to the jumping parameter. As mentioned before, this renders the optimal solution intractable because the separation principle does not hold. Thus, we assume a time-variant linear controller that is independent of the mode. This assumption allows us to express the considered finite-horizon cost function in closed form. However, because a naïve minimization of the cost function does not converge, we propose to minimize its upper bound. The main result of this paper consists in an iterative algorithm for computation of controller parameters based on the variational method that minimizes the proposed cost bound. Also, the convergence of the algorithm is shown.

In the next section, we formalize the considered problem and introduce some basic concepts. The main results of the paper are presented in Sec. III. We show a numerical example in Sec. IV and conclude the paper in Sec. V.

II. Problem Formulation and Basic Concepts

Consider a time-variant MJLS with dynamics

$$\begin{aligned} \underline{\boldsymbol{x}}_{k+1} &= \mathbf{A}_{k,\boldsymbol{\theta}_k}\underline{\boldsymbol{x}}_k + \mathbf{B}_{k,\boldsymbol{\theta}_k}\underline{u}_k + \mathbf{H}_{k,\boldsymbol{\theta}_k}\underline{\boldsymbol{w}}_k \ , \\ \underline{\boldsymbol{y}}_k &= \mathbf{C}_{k,\boldsymbol{\theta}_k}\underline{\boldsymbol{x}}_k + \mathbf{J}_{k,\boldsymbol{\theta}_k}\underline{\boldsymbol{v}}_k \ , \end{aligned} \tag{1}$$

where $\underline{\boldsymbol{x}}_k \in \mathbb{R}^n$ is the continuous-valued system state, $\underline{u}_k \in \mathbb{R}^m$ the control input, and $\underline{\boldsymbol{y}}_k \in \mathbb{R}^s$ the measurement. The discrete-valued mode $\boldsymbol{\theta}_k \in \{1, 2, \dots, M\}$, $M \in \mathbb{N} < \infty$ forms a Markov chain $\{\boldsymbol{\theta}_k\}$ with (time-variant) transition matrix $\mathbf{T}_k = [p_{k,ij}]_{M\times M}$, where $p_{k,ij} = \mathrm{P}(\boldsymbol{\theta}_{k+1} = j | \boldsymbol{\theta}_k = i)$ are the known transition probabilities. The matrix parameters of (1) are selected at each time step k from the time-variant sets $\{\mathbf{A}_{k,1}, \mathbf{B}_{k,1}, \mathbf{H}_{k,1}, \mathbf{C}_{k,1}, \mathbf{J}_{k,1}\}$, $\dots$, $\{\mathbf{A}_{k,M}, \mathbf{B}_{k,M}, \mathbf{H}_{k,M}, \mathbf{C}_{k,M}, \mathbf{J}_{k,M}\}$ according to the value of the mode $\boldsymbol{\theta}_k$. The initial state of the MJLS is given by a Gaussian mixture with means $\underline{\xi}_{0,i}$ and covariances $\mathbf{\Xi}_{0,i}$ weighted according to the initial distribution $\underline{\mu}_0$ of $\boldsymbol{\theta}_0$. Finally, $\underline{\boldsymbol{w}}_k \in \mathbb{R}^p$ and $\underline{\boldsymbol{v}}_k \in \mathbb{R}^q$ represent independent and identically distributed white Gaussian processes with zero mean and identity covariances. We can make this assumption without loss of generality because the matrices $\mathbf{H}_{k,\boldsymbol{\theta}_k}$ and $\mathbf{J}_{k,\boldsymbol{\theta}_k}$ can be selected in such a way that they model all possible covariances.

For system (1), we seek to find a control law that minimizes the finite-horizon quadratic cost function

$$\mathcal{J} = \mathrm{E}\left\{\underline{\boldsymbol{x}}_K^\top \mathbf{Q}_{K,\boldsymbol{\theta}_K}\underline{\boldsymbol{x}}_K + \sum_{k=0}^{K-1}\left[\underline{\boldsymbol{x}}_k^\top \mathbf{Q}_{k,\boldsymbol{\theta}_k}\underline{\boldsymbol{x}}_k + \underline{u}_k^\top \mathbf{R}_{k,\boldsymbol{\theta}_k}\underline{u}_k\right]\right\} , \tag{2}$$

where $K \in \mathbb{N} < \infty$ denotes the optimization horizon length, and the time-variant matrices $\mathbf{Q}_{k,\boldsymbol{\theta}_k}$ and $\mathbf{R}_{k,\boldsymbol{\theta}_k}$ are positive semidefinite and positive definite, respectively. The information available to the controller at time step k consists of the initial system state $(\underline{\xi}_{0,i}, \mathbf{\Xi}_{0,i}, \underline{\mu}_0)$ and the measurements $\underline{\boldsymbol{y}}_{0:k} = \{\underline{\boldsymbol{y}}_0, \underline{\boldsymbol{y}}_1, \dots, \underline{\boldsymbol{y}}_k\}$.

As mentioned above, the optimal controler for (1) that minimizes (2) without having access to the mode $\boldsymbol{\theta}_k$ is intractable. For this reason, we assume a mode-independent linear time-variant controller

$$\begin{aligned} \underline{\widehat{\boldsymbol{x}}}_{k+1} &= \mathbf{F}_k\underline{\widehat{\boldsymbol{x}}}_k + \mathbf{K}_k\underline{\boldsymbol{y}}_k \ , \\ \underline{u}_k &= \mathbf{L}_k\underline{\widehat{\boldsymbol{x}}}_k \ , \end{aligned} \tag{3}$$

where $\underline{\widehat{\boldsymbol{x}}}_k \in \mathbb{R}^n$ denotes the internal controller state.

Under this assumption, we can construct the closed-loop system dynamics by combining (1) and (3). It holds

$$\underline{\widetilde{\boldsymbol{x}}}_{k+1} = \widetilde{\mathbf{A}}_{k,\boldsymbol{\theta}_k}\underline{\widetilde{\boldsymbol{x}}}_k + \widetilde{\mathbf{H}}_{k,\boldsymbol{\theta}_k}\underline{\widetilde{\boldsymbol{w}}}_k \tag{4}$$

with

$$\underline{\widetilde{\boldsymbol{x}}}_k^\top = \begin{bmatrix}\underline{\boldsymbol{x}}_k^\top & \underline{\widehat{\boldsymbol{x}}}_k^\top\end{bmatrix} , \qquad \underline{\widetilde{\boldsymbol{w}}}_k^\top = \begin{bmatrix}\underline{\boldsymbol{w}}_k^\top & \underline{\boldsymbol{v}}_k^\top\end{bmatrix} ,$$

$$\widetilde{\mathbf{A}}_{k,\boldsymbol{\theta}_k} = \begin{bmatrix}\mathbf{A}_{k,\boldsymbol{\theta}_k} & \mathbf{B}_{k,\boldsymbol{\theta}_k}\mathbf{L}_k \\ \mathbf{K}_k\mathbf{C}_{k,\boldsymbol{\theta}_k} & \mathbf{F}_k\end{bmatrix} , \ \widetilde{\mathbf{H}}_{k,\boldsymbol{\theta}_k} = \begin{bmatrix}\mathbf{H}_{k,\boldsymbol{\theta}_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_k\mathbf{J}_{k,\boldsymbol{\theta}_k}\end{bmatrix} .$$

Next, we define the second moment of $\underline{\widetilde{\boldsymbol{x}}}_k$ as

$$\widetilde{\mathbf{X}}_{k,i} = \mathrm{E}\left\{\underline{\widetilde{\boldsymbol{x}}}_k\underline{\widetilde{\boldsymbol{x}}}_k^\top \mathbb{1}_{\boldsymbol{\theta}_k = i}\right\} ,$$

where the indicator function $\mathbb{1}_{\boldsymbol{\theta}_k = i}$ is 1 if $\boldsymbol{\theta}_k = i$ and 0 otherwise. The matrix $\widetilde{\mathbf{X}}_{k,i}$ is partitioned according to

$$\widetilde{\mathbf{X}}_{k,i} = \begin{bmatrix}\mathbf{X}_{k,i}^{(1)} & \mathbf{X}_{k,i}^{(12)} \\ (\mathbf{X}_{k,i}^{(12)})^\top & \mathbf{X}_{k,i}^{(2)}\end{bmatrix} .$$

The closed-loop second-moment dynamics of (4) can now be written as

$$\widetilde{\mathbf{X}}_{k+1,j} = \sum\nolimits_{i=1}^{M} p_{k,ij}\left[\widetilde{\mathbf{A}}_{k,i}\widetilde{\mathbf{X}}_{k,i}\widetilde{\mathbf{A}}_{k,i}^\top + \underline{\mu}_{k,i}\widetilde{\mathbf{H}}_{k,i}\widetilde{\mathbf{H}}_{k,i}^\top\right] ,$$

where $\mu_{k,i} = \mathrm{P}(\boldsymbol{\theta}_k = i)$. Similarly, we can rewrite (2) in terms of the closed-loop second-moment dynamics to

$$\mathcal{J}_t = \mathrm{E}\left\{\mathrm{tr}\left[\widetilde{\mathbf{Q}}_{K,\boldsymbol{\theta}_K}\widetilde{\mathbf{X}}_{K,\boldsymbol{\theta}_K} + \sum_{k=t}^{K-1}\widetilde{\mathbf{Q}}_{k,\boldsymbol{\theta}_k}\widetilde{\mathbf{X}}_{k,\boldsymbol{\theta}_k}\right]\right\} \tag{5}$$

for $t = 0, \dots, K$ with

$$\widetilde{\mathbf{Q}}_{K,\boldsymbol{\theta}_K} = \begin{bmatrix}\mathbf{Q}_{K,\boldsymbol{\theta}_K} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} , \ \widetilde{\mathbf{Q}}_{k,\boldsymbol{\theta}_k} = \begin{bmatrix}\mathbf{Q}_{k,\boldsymbol{\theta}_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_k^\top\mathbf{R}_{k,\boldsymbol{\theta}_k}\mathbf{L}_k\end{bmatrix} .$$

$$\overline{\mathbf{X}}_{k+1,j} = \sum_{i=1}^{M} p_{k,ij} \Big[\underline{\mu}_{k,i}\mathbf{H}_{k,i}\mathbf{H}_{k,i}^\top + \underline{\mu}_{k,i}\mathbf{K}_k\mathbf{J}_{k,i}\mathbf{J}_{k,i}^\top\mathbf{K}_k^\top + (\mathbf{A}_{k,i} - \mathbf{K}_k\mathbf{C}_{k,i})\overline{\mathbf{X}}_{k,i}(\mathbf{A}_{k,i} - \mathbf{K}_k\mathbf{C}_{k,i})^\top \\ +(\mathbf{A}_{k,i} - \mathbf{F}_k + \mathbf{B}_{k,i}\mathbf{L}_k - \mathbf{K}_k\mathbf{C}_{k,i})\underline{\mathbf{X}}_{k,i}(\mathbf{A}_{k,i} - \mathbf{F}_k + \mathbf{B}_{k,i}\mathbf{L}_k - \mathbf{K}_k\mathbf{C}_{k,i})^\top\Big] \ , \tag{9}$$

$$\underline{\mathbf{X}}_{k+1,j} = \sum_{i=1}^{M} p_{k,ij} \Big[\underline{\mu}_{k,i}\mathbf{H}_{k,i}\mathbf{H}_{k,i}^\top + \mathbf{K}_k\mathbf{C}_{k,i}\overline{\mathbf{X}}_{k,i}\mathbf{C}_{k,i}^\top\mathbf{K}_k^\top + (\mathbf{F}_k + \mathbf{K}_k\mathbf{C}_{k,i})\underline{\mathbf{X}}_{k,i}(\mathbf{F}_k + \mathbf{K}_k\mathbf{C}_{k,i})^\top\Big] \ , \tag{10}$$

$$\overline{\mathbf{P}}_{k,i} = \mathbf{Q}_{k,i} + \mathbf{L}_k^\top\mathbf{R}_{k,i}\mathbf{L}_k + (\mathbf{A}_{k,i} + \mathbf{B}_{k,i}\mathbf{L}_k)^\top\mathcal{E}_i(\overline{\mathbf{P}}_{k+1})(\mathbf{A}_{k,i} + \mathbf{B}_{k,i}\mathbf{L}_k) \\ + (\mathbf{A}_{k,i} - \mathbf{F}_k + \mathbf{B}_{k,i}\mathbf{L}_k - \mathbf{K}_k\mathbf{C}_{k,i})^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})(\mathbf{A}_{k,i} - \mathbf{F}_k + \mathbf{B}_{k,i}\mathbf{L}_k - \mathbf{K}_k\mathbf{C}_{k,i}) \tag{11}$$

$$\underline{\mathbf{P}}_{k,i} = \mathbf{L}_k^\top\mathbf{R}_{k,i}\mathbf{L}_k + \mathbf{L}_k^\top\mathbf{B}_{k,i}^\top\mathcal{E}_i(\overline{\mathbf{P}}_{k+1})\mathbf{B}_{k,i}\mathbf{L}_k + (\mathbf{F}_k - \mathbf{B}_{k,i}\mathbf{L}_k)^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})(\mathbf{F}_k - \mathbf{B}_{k,i}\mathbf{L}_k) \tag{12}$$

Finally, the following lemma presents a recursive formulation of (5).

Lemma 1 *For a fixed controller parameter sequence $\mathcal{G}_{0:K-1}$ with $\mathcal{G}_k = (\mathbf{F}_k, \mathbf{K}_k, \mathbf{L}_k)$, (5) can be written as a recursion according to*

$$\mathcal{J}_k = \sum_{i=1}^{M} \mathrm{tr}\left[\widetilde{\mathbf{P}}_{k,i}\widetilde{\mathbf{X}}_{k,i}\right] + \mu_{k,i}\omega_{k,i} \ , \tag{6}$$

where

$$\widetilde{\mathbf{P}}_{K,i} = \widetilde{\mathbf{Q}}_{K,i} \ , \quad \widetilde{\mathbf{P}}_{k,i} = \widetilde{\mathbf{Q}}_{k,i} + \widetilde{\mathbf{A}}_{k,i}^\top\mathcal{E}_i(\widetilde{\mathbf{P}}_{k+1})\widetilde{\mathbf{A}}_{k,i} \ , \tag{7}$$
$$\omega_{K,i} = 0 \ , \qquad \omega_{k,i} = \mathcal{E}_i(\omega_{k+1}) + \mathrm{tr}\left[\mathcal{E}_i(\widetilde{\mathbf{P}}_{k+1})\widetilde{\mathbf{H}}_{k,i}\widetilde{\mathbf{H}}_{k,i}^\top\right]$$

with $\mathcal{E}_i(\widetilde{\mathbf{P}}_{k+1}) = \sum_{j=1}^{M} p_{k,ij}\widetilde{\mathbf{P}}_{k+1,j}$.

Proof: Derivation of expression (6) for K is trivial. Let (6) hold for a time step $k+1$. Then, it follows

$$\begin{aligned}\mathcal{J}_k &= \sum_{i=1}^{M} \widetilde{\mathbf{Q}}_{k,i}\widetilde{\mathbf{X}}_{k,i} + \sum_{j=1}^{M} \mu_{k,i}p_{k,ij}\omega_{k+1,j} \\ &\quad + p_{k,ij}\mathrm{tr}\left[\widetilde{\mathbf{P}}_{k+1,j}\left(\widetilde{\mathbf{A}}_{k,i}\widetilde{\mathbf{X}}_{k,i}\widetilde{\mathbf{A}}_{k,i}^\top + \mu_{k,i}\widetilde{\mathbf{H}}_{k,i}\widetilde{\mathbf{H}}_{k,i}^\top\right)\right] \\ &= \sum_{i=1}^{M} \mathrm{tr}\left[(\widetilde{\mathbf{Q}}_{k,i} + \widetilde{\mathbf{A}}_{k,i}^\top\mathcal{E}_i(\widetilde{\mathbf{P}}_{k+1})\widetilde{\mathbf{A}}_{k,i})\widetilde{\mathbf{X}}_{k,i}\right] \\ &\quad + \mu_{k,i}\left[\mathcal{E}_i(\omega_{k+1}) + \mathrm{tr}\left[\mathcal{E}_i(\widetilde{\mathbf{P}}_{k+1})\right]\widetilde{\mathbf{H}}_{k,i}\widetilde{\mathbf{H}}_{k,i}^\top\right] \\ &= \sum_{i=1}^{M} \mathrm{tr}\left[\widetilde{\mathbf{P}}_{k,i}\widetilde{\mathbf{X}}_{k,i}\right] + \mu_{k,i}\omega_{k,i} \ .\end{aligned}$$

■

With these prerequisites, we are able to present the main results of this paper in the next section.

III. Main Results

In this section, we will present the main results of the paper. Our empirical evaluations have shown that a naïve minimization of (5) or (6), respectively, does not converge. A possible explanation for this issues is that the choice of non-optimal controller parameters $\mathcal{G}_k$ does not guarantee that $\widetilde{\mathbf{X}}_{k,i}$ and $\widetilde{\mathbf{P}}_{k,i}$ remain within the positive-definite cone, which makes the minimization of the cost function w.r.t. controller parameters ill-posed. For this reason, we will first present an upper bound for (6) in the next lemma. This bound will then be minimized in order to compute the controller parameters.

Lemma 2 *The recursion (6) can be bounded from above according to*

$$\mathcal{J}_k \leq \bar{\mathcal{J}}_k = \sum_{i=1}^{M} \mathrm{tr}\left[\widehat{\mathbf{P}}_{k,i}\widehat{\mathbf{X}}_{k,i}\right] + \mu_{k,i}\bar{\omega}_{k,i} \tag{8}$$

with

$$\widehat{\mathbf{X}}_{k,i} = \begin{bmatrix}\overline{\mathbf{X}}_{k,i} + \underline{\mathbf{X}}_{k,i} & \underline{\mathbf{X}}_{k,i} \\ \underline{\mathbf{X}}_{k,i} & \underline{\mathbf{X}}_{k,i}\end{bmatrix}, \widehat{\mathbf{P}}_{k,i} = \begin{bmatrix}\overline{\mathbf{P}}_{k,i} + \underline{\mathbf{P}}_{k,i} & -\underline{\mathbf{P}}_{k,i} \\ -\underline{\mathbf{P}}_{k,i} & \underline{\mathbf{P}}_{k,i}\end{bmatrix},$$

where the positive definite matrices $\overline{\mathbf{X}}_{k,i}$, $\underline{\mathbf{X}}_{k,i}$, $\overline{\mathbf{P}}_{k,i}$, and $\underline{\mathbf{P}}_{k,i}$ are given by (9)–(12), and

$$\bar{\omega}_{K,i} = 0 \ , \quad \bar{\omega}_{k,i} = \mathcal{E}_i(\bar{\omega}_{k+1}) + \mathrm{tr}\Big[\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{K}_k\mathbf{J}_{k,i}\mathbf{J}_{k,i}^\top\mathbf{K}_k^\top \\ +\mathcal{E}_i(\overline{\mathbf{P}}_{k+1} + \underline{\mathbf{P}}_{k+1})\mathbf{H}_{k,i}\mathbf{H}_{k,i}^\top\Big] \ .$$

Proof: We begin by defining

$$\underline{\mathbf{X}}_{k,i} = \mathbf{X}_{k,i}^{(2)} \ , \quad \overline{\mathbf{X}}_{k,i} = \mathbf{X}_{k,i}^{(1)} - \mathbf{X}_{k,i}^{(12)} - (\mathbf{X}_{k,i}^{(12)})^\top + \mathbf{X}_{k,i}^{(2)} \ ,$$
$$\underline{\mathbf{P}}_{k,i} = \mathbf{P}_{k,i}^{(2)} \ , \quad \overline{\mathbf{P}}_{k,i} = \mathbf{P}_{k,i}^{(1)} - \mathbf{P}_{k,i}^{(12)} - (\mathbf{P}_{k,i}^{(12)})^\top + \mathbf{P}_{k,i}^{(2)} \ , \tag{13}$$

where $\mathbf{P}_{k,i}^{(\alpha)}$ are partitions of $\widetilde{\mathbf{P}}_{k,i}$ with $\alpha \in \{1, 2, 12\}$. The expression for $\overline{\mathbf{X}}_{k,i}$ corresponds to the covariance $\mathrm{E}\left\{(\underline{\boldsymbol{x}}_k - \widehat{\underline{\boldsymbol{x}}}_k)(\underline{\boldsymbol{x}}_k - \widehat{\underline{\boldsymbol{x}}}_k)^\top\mathbb{1}_{\boldsymbol{\theta}_k=i}\right\}$ if we demand that the controller state $\widehat{\underline{\boldsymbol{x}}}_k$ is the unbiased first moment of the system state $\underline{\boldsymbol{x}}_k$. Analogously, the expression for $\overline{\mathbf{P}}_{k,i}$ corresponds to the covariance of the costate of $\underline{\boldsymbol{x}}_k$. Using (13), we can obtain (10) from the equation for $\mathbf{X}_{k,i}^{(2)}$ determined by (4). Then, plugging the equations for $\mathbf{X}_{k,i}^{(12)}$, $(\mathbf{X}_{k,i}^{(12)})^\top$, and $\mathbf{X}_{k,i}^{(2)}$ into the equation for $\overline{\mathbf{X}}_{k,i}$ in (13) yields (9). Analogously, we can obtain (12) and (11) from (7).

Next, we bound $\widetilde{\mathbf{X}}_{k,i}$ and $\widetilde{\mathbf{P}}_{k,i}$ from above according to

$$\widetilde{\mathbf{X}}_{k,i} \leq \begin{bmatrix}\overline{\mathbf{X}}_{k,i} + \underline{\mathbf{X}}_{k,i} & \underline{\mathbf{X}}_{k,i} \\ \underline{\mathbf{X}}_{k,i} & \underline{\mathbf{X}}_{k,i}\end{bmatrix}, \ \widetilde{\mathbf{P}}_{k,i} \leq \begin{bmatrix}\overline{\mathbf{P}}_{k,i} + \underline{\mathbf{P}}_{k,i} & -\underline{\mathbf{P}}_{k,i} \\ -\underline{\mathbf{P}}_{k,i} & \underline{\mathbf{P}}_{k,i}\end{bmatrix}. \tag{14}$$

To show this claim (here for $\widetilde{\mathbf{X}}_{k,i}$ only), we first use the substitutions $\mathbf{X}_{k,i}^{(2)} = \underline{\mathbf{X}}_{k,i}$ and $\mathbf{X}_{k,i}^{(1)} = \overline{\mathbf{X}}_{k,i} + \mathbf{X}_{k,i}^{(12)} + (\mathbf{X}_{k,i}^{(12)})^\top - \underline{\mathbf{X}}_{k,i}$ in $\widetilde{\mathbf{X}}_{k,i}$ that follow from (13) and take the Schur complement in $\underline{\mathbf{X}}_{k,i}$, which yields

$$\begin{aligned}&\overline{\mathbf{X}}_{k,i} + \mathbf{X}_{k,i}^{(12)} + (\mathbf{X}_{k,i}^{(12)})^\top - \underline{\mathbf{X}}_{k,i} - \mathbf{X}_{12}^{(k,i)}\underline{\mathbf{X}}_{k,i}^{-1}(\mathbf{X}_{12}^{(k,i)})^\top \\ &\leq \overline{\mathbf{X}}_{k,i} + \mathbf{X}_{k,i}^{(12)} + (\mathbf{X}_{k,i}^{(12)})^\top - \underline{\mathbf{X}}_{k,i} - \mathbf{X}_{k,i}^{(12)} - (\mathbf{X}_{k,i}^{(12)})^\top + \underline{\mathbf{X}}_{k,i} \\ &\leq \overline{\mathbf{X}}_{k,i} = \overline{\mathbf{X}}_{k,i} + \underline{\mathbf{X}}_{k,i} - \underline{\mathbf{X}}_{k,i}^\top\underline{\mathbf{X}}_{k,i}^{-1}\underline{\mathbf{X}}_{k,i} \ ,\end{aligned}$$

$$
\begin{aligned}
\mathcal{J}_k = \sum\nolimits_{i=1}^{M} \mathrm{tr}\big[&\mathbf{Q}_{i,k}(\overline{\mathbf{X}}_{i,k}+\underline{\mathbf{X}}_{i,k}) + \mathbf{L}_k^\top \mathbf{R}_{i,k}\mathbf{L}_k\underline{\mathbf{X}}_{i,k} + \mathbf{A}_{i,k}^\top \mathcal{E}_i(\overline{\mathbf{P}}_{k+1}+\underline{\mathbf{P}}_{k+1})\mathbf{A}_{i,k}(\overline{\mathbf{X}}_{i,k}+\underline{\mathbf{X}}_{i,k}) \\
&-2\mathbf{A}_{i,k}^\top \mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{K}_k\mathbf{C}_{i,k}(\overline{\mathbf{X}}_{i,k}+\underline{\mathbf{X}}_{i,k}) + \mathbf{C}_{i,k}^\top\mathbf{K}_k^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{K}_k\mathbf{C}_{i,k}(\overline{\mathbf{X}}_{i,k}+\underline{\mathbf{X}}_{i,k}) + 2\mathbf{A}_{i,k}^\top\mathcal{E}_i(\overline{\mathbf{P}}_{k+1}+\underline{\mathbf{P}}_{k+1})\mathbf{B}_{i,k}\mathbf{L}_k\underline{\mathbf{X}}_{i,k} \\
&-2\mathbf{C}_{i,k}^\top\mathbf{K}_k^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{B}_{i,k}\mathbf{L}_k\underline{\mathbf{X}}_{i,k} - 2\mathbf{A}_{i,k}^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{F}_k\underline{\mathbf{X}}_{i,k} + 2\mathbf{C}_{i,k}^\top\mathbf{K}_k^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{F}_k\underline{\mathbf{X}}_{i,k} + \mathbf{F}_k^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1}\mathbf{F}_k\underline{\mathbf{X}}_{i,k}) \\
&+\mathbf{L}_k^\top\mathbf{B}_{i,k}^\top\mathcal{E}_i(\overline{\mathbf{P}}_{k+1}+\underline{\mathbf{P}}_{k+1})\mathbf{B}_{i,k}\mathbf{L}_k\underline{\mathbf{X}}_{i,k} - 2\mathbf{F}_k^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{B}_{i,k}\mathbf{L}_k\underline{\mathbf{X}}_{i,k} + \mu_{i,k}\mathcal{E}_i(\overline{\mathbf{P}}_{k+1}+\underline{\mathbf{P}}_{k+1})\mathbf{H}_{i,k}\mathbf{H}_{i,k}^\top \\
&+\mu_{i,k}\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{K}_k\mathbf{J}_{i,k}\mathbf{J}_{i,k}^\top\mathbf{K}_k^\top\big] + \mu_{i,k}\mathcal{E}_i(\bar{\omega}_{k+1}) \ ,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\mathcal{J}_k = \sum\nolimits_{i=1}^{M} \mathrm{tr}\big[&\mathbf{Q}_{i,k}(\overline{\mathbf{X}}_{i,k}+\underline{\mathbf{X}}_{i,k}) + \mathbf{A}_{i,k}^\top\mathcal{E}_i(\overline{\mathbf{P}}_{k+1}+\underline{\mathbf{P}}_{k+1})\mathbf{A}_{i,k}(\overline{\mathbf{X}}_{i,k}+\underline{\mathbf{X}}_{i,k}) + \mu_{i,k}\mathcal{E}_i(\overline{\mathbf{P}}_{k+1}+\underline{\mathbf{P}}_{k+1})\mathbf{H}_{i,k}\mathbf{H}_{i,k}^\top\big] \\
&+\mu_{i,k}\mathcal{E}_i(\bar{\omega}_{k+1}) + 2\underline{\rho}_k^\top\mathrm{vec}(\mathbf{K}_k) + 2\underline{\gamma}_k^\top\mathrm{vec}(\mathbf{L}_k) + \mathrm{vec}(\mathbf{K}_k)^\top\mathbf{\Gamma}_k\mathrm{vec}(\mathbf{K}_k) + \mathrm{vec}(\mathbf{L}_k)^\top\mathbf{\Phi}_k\mathrm{vec}(\mathbf{L}_k) - 2\mathrm{vec}(\mathbf{F}_k)^\top\underline{\psi}_k \\
&+2\mathrm{vec}(\mathbf{K}_k)^\top\mathbf{\Upsilon}_k^\top\mathrm{vec}(\mathbf{L}_k) + 2\mathrm{vec}(\mathbf{F}_k)^\top\mathbf{\Psi}_{k,2}\mathrm{vec}(\mathbf{K}_k) - 2\mathrm{vec}(\mathbf{F}_k)^\top\mathbf{\Sigma}_k\mathrm{vec}(\mathbf{L}_k) + \mathrm{vec}(\mathbf{F}_k)^\top\mathbf{\Lambda}_k\mathrm{vec}(\mathbf{F}_k) \ .
\end{aligned} \tag{17}
$$

---

where we used the inequality $\mathbf{X}\mathbf{Y}^{-1}\mathbf{X}^\top \geq \mathbf{X}+\mathbf{X}^\top-\mathbf{Y}$. Then, $\widetilde{\mathbf{X}}_{k,i} \leq \widehat{\mathbf{X}}_{k,i}$ holds because reversing the Schur complement preserves the partial ordering property of positive definiteness. Finally, the bounds $\widetilde{\mathbf{X}}_{k,i} \leq \widehat{\mathbf{X}}_{k,i}$ and $\widetilde{\mathbf{P}}_{k,i} \leq \widehat{\mathbf{P}}_{k,i}$ used in (6) yield the result of the lemma. ■

Now that we have the result from Lemma 2, we will present the necessary optimality conditions for the considered problem in the following theorem. This result will be used in the iterative algorithm that computes the controller parameters $\mathcal{G}_{0:K-1}$.

Theorem 1 *Let the sequence $\mathcal{G}_{0:K-1}$ minimize the bound* (8). *Then, it holds*

$$
\begin{bmatrix} \mathbf{\Lambda}_k & \mathbf{\Gamma}_k & -\mathbf{\Sigma}_k \\ \mathbf{\Gamma}_k^\top & \mathbf{\Psi}_k & \mathbf{\Upsilon}_k \\ -\mathbf{\Sigma}_k^\top & \mathbf{\Upsilon}_k^\top & \mathbf{\Phi}_k \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\mathbf{F}_k) \\ \mathrm{vec}(\mathbf{K}_k) \\ \mathrm{vec}(\mathbf{L}_k) \end{bmatrix} + \begin{bmatrix} -\underline{\psi}_k \\ \underline{\rho}_k \\ \underline{\gamma}_k \end{bmatrix} = \underline{0} \ , \tag{15}
$$

where $\mathrm{vec}(\cdot)$ *denotes the vectorization operator and*

$$
\begin{aligned}
\mathbf{\Lambda}_k &= \sum\nolimits_{i=1}^{M}\left[\underline{\mathbf{X}}_{i,k}\otimes\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\right] \ , \\
\mathbf{\Psi}_k &= \sum\nolimits_{i=1}^{M}\big[(\mu_{i,k}\mathbf{J}_{i,k}\mathbf{J}_{i,k}^\top + \mathbf{C}_{i,k}(\overline{\mathbf{X}}_{i,k}+\underline{\mathbf{X}}_{i,k})\mathbf{C}_{i,k}^\top) \\
&\qquad\qquad \otimes\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\big] \ , \\
\mathbf{\Gamma}_k &= \sum\nolimits_{i=1}^{M}\left[\underline{\mathbf{X}}_{i,k}\mathbf{C}_{i,k}^\top\otimes\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\right] \ , \\
\mathbf{\Phi}_k &= \sum\nolimits_{i=1}^{M}\left[\underline{\mathbf{X}}_{i,k}\otimes(\mathbf{R}_{i,k}+\mathbf{B}_{i,k}^\top\mathcal{E}_i(\overline{\mathbf{P}}_{k+1}+\underline{\mathbf{P}}_{k+1}))\mathbf{B}_{i,k}\right] , \\
\mathbf{\Sigma}_k &= \sum\nolimits_{i=1}^{M}\left[\underline{\mathbf{X}}_{i,k}\otimes\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{B}_{i,k}\right] \ , \\
\mathbf{\Upsilon}_k &= -\sum\nolimits_{i=1}^{M}\left[\underline{\mathbf{X}}_{i,k}\mathbf{C}_{i,k}^\top\otimes\mathbf{B}_{i,k}^\top\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\right] \ , \\
\underline{\psi}_k &= \mathrm{vec}\left(\sum\nolimits_{i=1}^{M}\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{A}_{i,k}\underline{\mathbf{X}}_{i,k}\right) \ , \\
\underline{\rho}_k &= -\mathrm{vec}\left(\sum\nolimits_{i=1}^{M}\mathcal{E}_i(\underline{\mathbf{P}}_{k+1})\mathbf{A}_{i,k}(\overline{\mathbf{X}}_{i,k}+\underline{\mathbf{X}}_{i,k})\mathbf{C}_{i,k}^\top\right) \ , \\
\underline{\gamma}_k &= \mathrm{vec}\left(\sum\nolimits_{i=1}^{M}\mathbf{B}_{i,k}^\top\mathcal{E}_i(\overline{\mathbf{P}}_{k+1}+\underline{\mathbf{P}}_{k+1})\mathbf{A}_{i,k}\underline{\mathbf{X}}_{i,k}\right) \ ,
\end{aligned}
$$

where $\otimes$ denotes the Kronecker operator.

Proof: First, evaluation of (8) yields (16). Then, using the identities $\mathrm{tr}\left[\mathbf{A}^\top\mathbf{B}\right] = \mathrm{vec}(\mathbf{A})^\top\mathrm{vec}(\mathbf{B})$ and $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^\top\otimes\mathbf{A})\mathrm{vec}(\mathbf{B})$, we can rewrite (16) according to (17), which in tern can be written as a quadratic function of the stacked vector $\begin{bmatrix}\mathrm{vec}(\mathbf{F}_k)^\top & \mathrm{vec}(\mathbf{K}_k)^\top & \mathrm{vec}(\mathbf{L}_k)^\top\end{bmatrix}^\top$. Finally, evaluation of the necessary optimality condition for $\overline{\mathcal{J}}_k$ w.r.t. this vector yields (15). ■

With the result from Theorem 1, the algorithm for computation of the controller parameters $\mathcal{G}_{0:K-1}$ can be formulated as given in Fig. 1. A reference implementation of this algorithm is available on GitHub [23].

- *Step 1:* Set the counter $\eta = 0$, pick a random sequence $\mathcal{G}_{0:K-1}^{[\eta]}$, and initialize $\bar{\mathcal{J}}_0^{[0]} = \infty$.
- *Step 2:* Compute the sequence $\overline{\mathbf{X}}_{k,i}^{[\eta]}$ and $\underline{\mathbf{X}}_{k,i}^{[\eta]}$ for $k = 1, 2, \ldots, K$ initialized with $\overline{\mathbf{X}}_{0,i}^{[\eta]} = \mathbf{\Xi}_{0,i}$ and $\underline{\mathbf{X}}_{0,i}^{[\eta]} = \underline{\xi}_{0,i}\underline{\xi}_{0,i}^\top$ using (9), (10), and $\mathcal{G}_{0:K-1}^{[\eta]}$.
- *Step 3:* Initialize $\overline{\mathbf{P}}_{K,i}^{[\eta+1]} = \mathbf{Q}_{K,i}$, $\underline{\mathbf{P}}_{K,i}^{[\eta+1]} = \mathbf{0}$ and set $k = K-1$.
- *Step 4:* Compute $\mathcal{G}_k^{[\eta+1]} = (\mathbf{F}_k^{[\eta+1]}, \mathbf{K}_k^{[\eta+1]}, \mathbf{L}_k^{[\eta+1]})$ using (15). Compute $\overline{\mathbf{P}}_{k,i}^{[\eta+1]}$ and $\underline{\mathbf{P}}_{k,i}^{[\eta+1]}$ using (11) and (12). If $k > 0$, set $k = k-1$ and return to *Step 4*. Otherwise, proceed to *Step 5*.
- *Step 5:* Evaluate $\bar{\mathcal{J}}_0^{[\eta+1]}$. If $\bar{\mathcal{J}}_0^{[\eta]} - \bar{\mathcal{J}}_0^{[\eta+1]}$ is sufficiently small, stop the algorithm. Otherwise, set $\eta = \eta + 1$ and return to *Step 2*.

Fig. 1. Algorithm for computation of the controller parameters.

Faster convergence of the algorithm can be achieved if in *Step 1*, we initialize $\mathcal{G}_{0:K-1}$ with the infinite-horizon controller parameters $\mathcal{G}_\infty$ from [19]. Furthermore, the proposed algorithm can be implemented in a moving-horizon framework for a better performance. In order to increase the convergence speed, we propose to make the initialization in *Step 1* for $k = 1, \ldots, K-1$ with already computed parameters $\mathcal{G}_{1:K-1}$ from the previous time step and $\mathcal{G}_\infty$ for $k = K$. Finally, we conclude this section with the proof of convergence of the proposed algorithm.

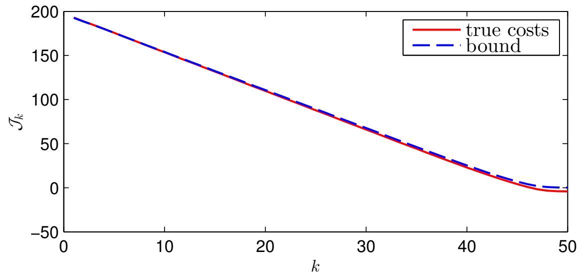

Fig. 2. True costs and the proposed bound for converged sequence $\mathcal{G}_{0:K-1}$ with $\mathbf{T}_{[1]}$.

Theorem 2 *The proposed algorithm generates a sequence of monotonically decreasing costs, i.e., $\bar{\mathcal{J}}_k^{[\eta]} \leq \bar{\mathcal{J}}_k^{[\eta-1]}$ for $k = 0, 1, \ldots, K$, which implies that $\mathcal{G}_{0:K-1}^* = \lim_{\eta\to\infty} \mathcal{G}_{0:K-1}^{[\eta]}$ exists.*

Proof: Define the sequence of controller parameters $\mathcal{G}_{0:K-1}^{[\eta,t]} = \{(\mathbf{F}_0^{[\eta-1]}, \mathbf{K}_0^{[\eta-1]}, \mathbf{L}_0^{[\eta-1]}), \ldots, (\mathbf{F}_{t-1}^{[\eta-1]}, \mathbf{K}_{t-1}^{[\eta-1]}, \mathbf{L}_{t-1}^{[\eta-1]}), (\mathbf{F}_t^{[\eta]}, \mathbf{K}_t^{[\eta]}, \mathbf{L}_t^{[\eta]}), \ldots, (\mathbf{F}_{K-1}^{[\eta]}, \mathbf{K}_{K-1}^{[\eta]}, \mathbf{L}_{K-1}^{[\eta]})\}$. In this sequence, the controller parameters have been updated for time steps $k \geq t$ and the others have not. In the sequel, we will evaluate the cost difference $\bar{\mathcal{J}}_k^{[\eta,k]} - \bar{\mathcal{J}}_k^{[\eta,k+1]}$ for $k = 0, 1, \ldots, K$, in order to prove the claim of Theorem 2, where $\bar{\mathcal{J}}_k^{[\eta,k]}$ is computed using the sequence $\mathcal{G}_{0:K-1}^{[\eta,k]}$ and $\bar{\mathcal{J}}_k^{[\eta,k+1]}$ using $\mathcal{G}_{0:K-1}^{[\eta,k+1]}$. Because we compute $\mathcal{G}_{0:K-1}$ backwards from $K-1$ to 0, we have that the kth element gets updated from $\mathcal{G}_k^{[\eta-1]}$ to $\mathcal{G}_k^{[\eta]}$, while the other elements in $\mathcal{G}_{0:K-1}^{[\eta,k]}$ remain unchanged, i.e., $\mathcal{G}_{0:k-1}^{[\eta,k]} = \mathcal{G}_{0:k-1}^{[\eta,k+1]}$ and $\mathcal{G}_{k+1:K-1}^{[\eta,k]} = \mathcal{G}_{k+1:K-1}^{[\eta,k+1]}$. Thus, $\overline{\mathbf{P}}_{k+1:K}^{[\eta]}$ and $\underline{\mathbf{P}}_{k+1:K}^{[\eta]}$ computed using $\mathcal{G}_{0:K}^{[\eta,k]}$ and $\mathcal{G}_{0:K}^{[\eta,k+1]}$ are equal. This also implies that $\bar{\omega}_{k:K}^{[\eta]}$ computed using the two sequences $\mathcal{G}_{0:K}^{[\eta,k]}$ and $\mathcal{G}_{0:K}^{[\eta,k+1]}$ are also equal and cancel out in $\bar{\mathcal{J}}_k^{[\eta]} - \bar{\mathcal{J}}_k^{[\eta-1]}$. Additionally, we also have the equivalence of $\overline{\mathbf{X}}_{0:k}^{[\eta-1]}$ and $\underline{\mathbf{X}}_{0:k}^{[\eta-1]}$ computed using the sequences $\mathcal{G}_{0:K-1}^{[\eta,k]}$ and $\mathcal{G}_{0:K-1}^{[\eta,k+1]}$. Consequently, the only difference in the elements of $\mathcal{G}_{0:K-1}^{[\eta,k]}$ and $\mathcal{G}_{0:K-1}^{[\eta,k+1]}$ is that $\mathcal{G}_k^{[\eta,k]} = (\mathbf{F}_k, \mathbf{K}_k, \mathbf{L}_k)$ satisfies (15), while $\mathcal{G}_k^{[\eta,k+1]} = (\mathbf{F}'_k, \mathbf{K}'_k, \mathbf{L}'_k)$ does not. With the preceding observations and according to (17), we can evaluate the cost difference $\mathcal{J}_k^{[\eta,k]} - \mathcal{J}_k^{[\eta,k+1]}$ to

$$\mathcal{J}_k^{[\eta,k]} - \mathcal{J}_k^{[\eta,k+1]} = 2 \begin{bmatrix} \mathrm{vec}(\mathbf{F}_k) - \mathrm{vec}(\mathbf{F}'_k) \\ \mathrm{vec}(\mathbf{K}_k) - \mathrm{vec}(\mathbf{K}'_k) \\ \mathrm{vec}(\mathbf{L}_k) - \mathrm{vec}(\mathbf{L}'_k) \end{bmatrix}^\top \begin{bmatrix} -\underline{\psi}_k \\ \underline{\rho}_k \\ \underline{\gamma}_k \end{bmatrix}$$
$$+ \begin{bmatrix} \mathrm{vec}(\mathbf{F}_k) \\ \mathrm{vec}(\mathbf{K}_k) \\ \mathrm{vec}(\mathbf{L}_k) \end{bmatrix}^\top \begin{bmatrix} \mathbf{\Lambda}_k & \mathbf{\Gamma}_k & -\mathbf{\Sigma}_k \\ \mathbf{\Gamma}_k^\top & \mathbf{\Psi}_k & \mathbf{\Upsilon}_k \\ -\mathbf{\Sigma}_k^\top & \mathbf{\Upsilon}_k^\top & \mathbf{\Phi}_k \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\mathbf{F}_k) \\ \mathrm{vec}(\mathbf{K}_k) \\ \mathrm{vec}(\mathbf{L}_k) \end{bmatrix}$$
$$- \begin{bmatrix} \mathrm{vec}(\mathbf{F}'_k) \\ \mathrm{vec}(\mathbf{K}'_k) \\ \mathrm{vec}(\mathbf{L}'_k) \end{bmatrix}^\top \begin{bmatrix} \mathbf{\Lambda}_k & \mathbf{\Gamma}_k & -\mathbf{\Sigma}_k \\ \mathbf{\Gamma}_k^\top & \mathbf{\Psi}_k & \mathbf{\Upsilon}_k \\ -\mathbf{\Sigma}_k^\top & \mathbf{\Upsilon}_k^\top & \mathbf{\Phi}_k \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\mathbf{F}'_k) \\ \mathrm{vec}(\mathbf{K}'_k) \\ \mathrm{vec}(\mathbf{L}'_k) \end{bmatrix} .$$

where $\mathbf{\Psi}_k, \mathbf{\Phi}_k, \mathbf{\Lambda}_k, \mathbf{\Upsilon}_k, \mathbf{\Gamma}_k, \mathbf{\Sigma}_k, \underline{\psi}_k, \underline{\rho}_k$, and $\underline{\gamma}_k$ are calculated using $\overline{\mathbf{X}}_{k,i}^{[\eta-1]}, \underline{\mathbf{X}}_{k,i}^{[\eta-1]}, \overline{\mathbf{P}}_{i,k+1}^{[\eta]}$, and $\underline{\mathbf{P}}_{k+1,i}^{[\eta]}$. After substituting $\begin{bmatrix} -\underline{\psi}_k^\top & \underline{\rho}_k^\top & \underline{\gamma}_k^\top \end{bmatrix}^\top$ with

$$- \begin{bmatrix} \mathbf{\Lambda}_k & \mathbf{\Gamma}_k & -\mathbf{\Sigma}_k \\ \mathbf{\Gamma}_k^\top & \mathbf{\Psi}_k & \mathbf{\Upsilon}_k \\ -\mathbf{\Sigma}_k^\top & \mathbf{\Upsilon}_k^\top & \mathbf{\Phi}_k \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\mathbf{F}_k) \\ \mathrm{vec}(\mathbf{K}_k) \\ \mathrm{vec}(\mathbf{L}_k) \end{bmatrix}$$

according to (15), we finally obtain

$$\mathcal{J}_k^{[\eta,k]} - \mathcal{J}_k^{[\eta,k+1]} = - \begin{bmatrix} \mathrm{vec}(\mathbf{F}_k) - \mathrm{vec}(\mathbf{F}'_k) \\ \mathrm{vec}(\mathbf{K}_k) - \mathrm{vec}(\mathbf{K}'_k) \\ \mathrm{vec}(\mathbf{L}_k) - \mathrm{vec}(\mathbf{L}'_k) \end{bmatrix}^\top$$
$$\times \begin{bmatrix} \mathbf{\Lambda}_k & \mathbf{\Gamma}_k & -\mathbf{\Sigma}_k \\ \mathbf{\Gamma}_k^\top & \mathbf{\Psi}_k & \mathbf{\Upsilon}_k \\ -\mathbf{\Sigma}_k^\top & \mathbf{\Upsilon}_k^\top & \mathbf{\Phi}_k \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\mathbf{F}_k) - \mathrm{vec}(\mathbf{F}'_k) \\ \mathrm{vec}(\mathbf{K}_k) - \mathrm{vec}(\mathbf{K}'_k) \\ \mathrm{vec}(\mathbf{L}_k) - \mathrm{vec}(\mathbf{L}'_k) \end{bmatrix} ,$$

which is clearly not positive. The claim $\bar{\mathcal{J}}_k^{[\eta]} \leq \bar{\mathcal{J}}_k^{[\eta-1]}$ for $k = 0, \ldots, K$ then follows by induction. ■

IV. NUMERICAL EXAMPLE

In this section, we demonstrate the proposed algorithm in a numerical example, where we compare it with our infinite-horizon controller from [19] and the optimal linear controller from [7] that requires mode feedback. To this end, the parameters of (1) are assumed to be time-invariant and are chosen to

$$\mathbf{A}_1 = \begin{bmatrix} 1.2 & 1.2 \\ 0 & 1 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 1 & 1.3 \\ 0 & 1 \end{bmatrix}, \ \mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1.4 \end{bmatrix},$$
$$\mathbf{H}_1 = \mathbf{H}_2 = 0.2^2\mathbf{I}, \ \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ \mathbf{C}_2 = \begin{bmatrix} 0.8 & 0 \end{bmatrix},$$
$$\mathbf{J}_1 = \mathbf{J}_2 = 0.1^2\mathbf{I}, \ \mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{I}, \ \mathbf{R}_1 = \mathbf{R}_2 = 1,$$

and the initial conditions are set to $\underline{\mu}_0 = [1\ 0]^\top, \underline{\xi}_{0,1} = [5\ 0]^\top$, and $\mathbf{\Xi}_{0,1} = 0.1^2\mathbf{I}$. Furthermore, the two transition matrices

$$\mathbf{T}_{[1]} = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{T}_{[2]} = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}$$

are investigated. The simulation time is set to $K = 50$ and we perform 10^4 Monte Carlo runs. The proposed controller is computed only once for the entire simulation time, i.e., we do not implement it in a receding horizon framework. The simulation results are depicted in Table II where the true normalized costs $\mathcal{J}_0/K$ can be seen. The simulation results show that the time-variant controller performs slightly better than the time-invariant controller from [19]. Also, the performance decrease is small compared to the optimal controller from [7] that requires mode feedback. Further, an example run with $\mathbf{T}_{[1]}$ is shown in Fig. 3. The trajectories depicted in this figure are consistent with the results in

	$\mathbf{T}_{[1]}$	$\mathbf{T}_{[2]}$
Optimal [7]	1.0716	1.0768
Time-invariant [19]	1.0883	1.1205
Proposed	1.0716	1.0827

TABLE II

MEAN COSTS OF THE SIMULATED CONTROLLERS.

Table II. Furthermore, it can be seen that the time-variant controller performs better than its time-invariant counterpart within approx. the first 10 simulation time steps. This observation motivates an implementation of the proposed algorithm in a receding horizon framework. Finally, Fig. 2 shows the expected costs (2) and its bound (8) for the converged sequence $\mathcal{G}_{0:K-1}$. It can be seen that the bound is very close and thus the corresponding controller parameter sequence will be almost optimal in the linear sense.

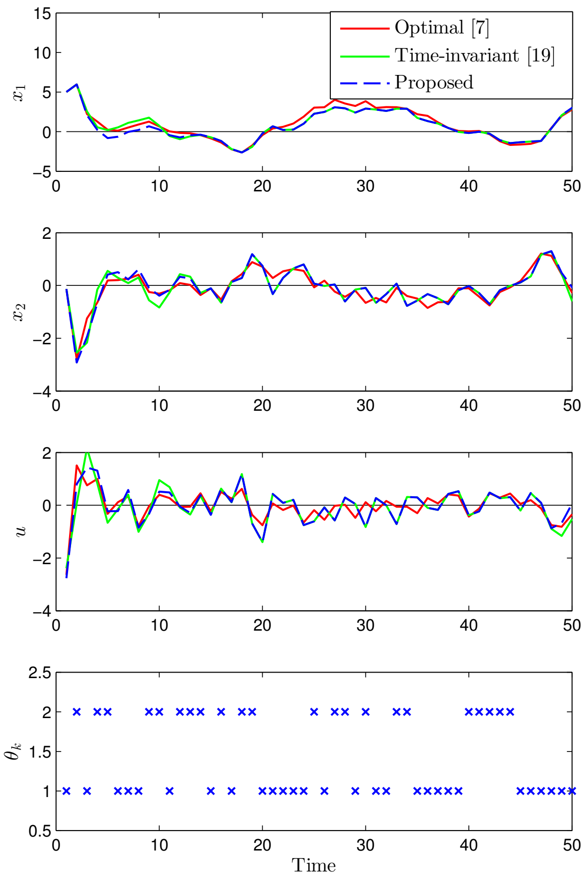

Fig. 3. Example run with $\mathbf{T}_{[1]}$.

V. CONCLUSION

In this paper, we proposed an algorithm for computation of the parameters of a time-variant linear controller for finite-horizon dynamic compensation of MJLS without mode observation and showed its convergence. To this end, we assumed a linear controller that consists of an estimator and a regulator. However, because the separation between the two components does not hold and a naïve computation of the controller parameters does not converge, we proposed to minimize an upper bound on the true costs of the closed-loop system. In our simulations, the time-variant controller, whose parameters were computed using the proposed algorithm, showed better performance than its time-invariant counterpart from [19]. Also, the performance loss is small compared to the optimal controller from [7] that uses mode feedback.

REFERENCES

[1] J. B. R. do Val and T. Başar, "Receding Horizon Control of Jump Linear Systems and a Macroeconomic Policy Problem," *Journal of Economic Dynamics and Control*, vol. 23, no. 8, 1999.

[2] R. J. Elliott, T. K. Siu, L. Chan, and J. W. Lau, "Pricing Options Under a Generalized Markov-Modulated Jump-Diffusion Model," *Stochastic Analysis and Applications*, vol. 25, no. 4, pp. 821–843, 2007.

[3] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A Survey of Recent Results in Networked Control Systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[4] J. Fischer, A. Hekler, M. Dolgov, and U. D. Hanebeck, "Optimal Sequence-Based LQG Control over TCP-like Networks Subject to Random Transmission Delays and Packet Losses," in *Proceedings of the 2013 American Control Conference (ACC 2013)*, Washington D. C., USA, Jun. 2013.

[5] A. N. Vargas, E. F. Costa, and J. B. R. do Val, "On the Control of Markov Jump Linear Systems with no Mode Observation: Application to a DC Motor Device," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 10, pp. 1136–1150, 2013.

[6] R. Oliveira, A. Vargas, J. do Val, and P. Peres, "Mode-Independent H_2-Control of a DC Motor Modeled as a Markov Jump Linear System," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 5, pp. 1915–1919, Sept 2014.

[7] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*, 1st ed. London: Springer, 2005.

[8] N. N. Krasovskii and E. A. Lidskii, "Analytic Design of Controller in Systems with Random Attributes - Part I," *Automation and Remote Control*, vol. 22, pp. 1021–1025, 1961.

[9] Y. Ji and H. J. Cchizeck, "Controllability, Observability and Discrete-time Markovian Jump Linear Quadratic Control," *International Journal of Control*, vol. 48, no. 2, pp. 481–498, 1988.

[10] Y. Ji, H. J. Chizeck, X. Feng, and K. A. Loparo, "Stability and Control of Discrete-time Jump Linear Systems," *Control Theory and Advanced Technology*, vol. 7, no. 2, pp. 247–270, 1991.

[11] B. Griffiths and K. A. Loparo, "Optimal Control of Jump Linear Gaussian Systems," *International Journal of Control*, vol. 43, no. 4, pp. 792–819, 1983.

[12] F. Casiello and K. A. Loparo, "Optimal Control of Unknown Parameter Systems," *IEEE Transactions on Automatic Control*, vol. 34, 1989.

[13] A. N. Vargas, L. Acho, G. Pujol, E. F. Costa, J. Y. Ishihara, and J. B. R. do Val, "Output Feedback of Markov Jump Linear Systems with no Mode Observation: An Automotive Throttle Application," *International Journal of Robust and Nonlinear Control*, 2015.

[14] A. N. Vargas, D. C. Bortolin, E. F. Costa, and J. B. R. do Val, "Gradient-based Optimization Techniques for the Design of Static Controllers for Markov Jump Linear Systems with Unobservable Modes," *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, 2014.

[15] J. B. R. do Val, J. C. Geromel, and A. P. C. Gonçalves, "The H_2-Control for Jump Linear Systems: Cluster Observations of the Markov State," *Automatica*, vol. 38, no. 2, pp. 343 – 349, 2002.

[16] M. Dolgov and U. D. Hanebeck, "Infinite-horizon Linear Optimal Control of Markov Jump Systems without Mode Observation via State Feedback," *arXiv preprint: Systems and Control (cs.SY)*, Jul. 2015.

[17] ——, "Static Output-Feedback Control of Markov Jump Linear Systems without Mode Observation," Sept. 2015.

[18] A. R. Fioravanti, A. P. Goncalves, and J. C. Geromel, "Discrete-time H_∞ output feedback for Markov jump systems with uncertain transition probabilities," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 8, pp. 894–902, 2013.

[19] M. Dolgov and U. D. Hanebeck, "Dynamic Output-Feedback Control of Markov Jump Linear Systems without Mode Observation (to appear)," in *Proceedings of the 2016 European Control Conference (ECC 2016)*, Aalborg, Denmark, Jun. 2016.

[20] G. Pan and Y. Bar-Shalom, "Stabilization of Jump Linear Gaussian Systems without Mode Observations," *International Journal of Control*, vol. 64, no. 4, 1996.

[21] L. Campo and Y. Bar-Shalom, "Control of Discrete-time Hybrid Stochastic Systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 10, pp. 1522–1527, Oct 1992.

[22] O. Costa and E. Tuesta, "Finite horizon quadratic optimal control and a separation principle for markovian jump linear systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 10, 2003.

[23] M. Dolgov, "Linear Control of MJLS without Mode Observation," https://github.com/Mxttak/MJLS, Mar. 2016.