

Stochastic Nonlinear Model Predictive Control Based on Progressive Density Simplification

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Abstract—Increasing demand for Nonlinear Model Predictive Control with the ability to handle highly noise-corrupted systems has recently given rise to stochastic control approaches. Besides providing high-quality results within a noisy environment, these approaches have one problem in common, namely a high computational demand and, as a consequence, generally a short prediction horizon. In this paper, we propose to reduce the computational complexity of prediction and value function evaluation within the control horizon by simplifying the system progressively down to the deterministic case. Approximation of occurring probability densities by a specific representation, the deterministic Dirac mixture density, with a decreasing resolution (i.e., approximation quality) leads via natural decomposition to a point estimate and thus, can be treated in a deterministic manner. Hence, calculation of the remaining time steps requires considerably less computation time.

I. INTRODUCTION

In Model Predictive Control (MPC), also referred to as Receding or Rolling Horizon Control, the decision on the control input is not only based on the current system state, but also incorporates information about future system states, which often results in higher quality of control. If no measurements are regarded during planning, an open-loop optimal control problem is solved according to a value function over a prediction horizon [1]. This results in a decision for an optimal control input, which is applied to the system in terms of a closed-loop control.

Linear system models do not generally reflect the real-world system behavior in an adequate way. As many applications have growing requirements on the control quality, the widely used MPC for linear system models is often not sufficient [2]. Hence, it is strongly desirable that the controller incorporates nonlinear system models.

Though advanced methods for nonlinear systems have been developed solving the open-loop optimization [3], most approaches do not explicitly consider occurring uncertainties in the control decision [4]. In an early approach, the Extended Kalman Filter was introduced to Nonlinear Model Predictive Control (NMPC) [5]. Later on, more sophisticated methods used approximations of the occurring probability densities [6], [7], [8] and, recently, the Particle Filter was applied in an NMPC framework [9]. Newly arising fields of application, such as Networked Control Systems [10], demand a high quality of control, while acting within an uncertain environment. In order to ensure robustness, it is important

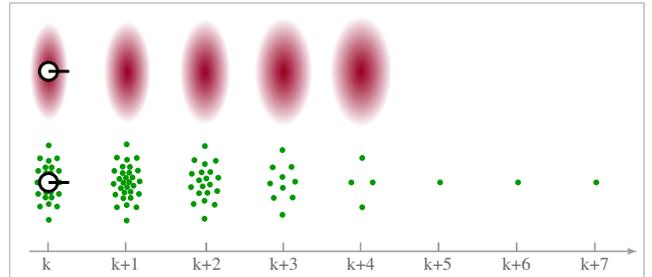


Fig. 1. Comparison of the prediction horizon for NMPC based on an Extended Kalman Filter (red gradient) and for the proposed stochastic NMPC based on progressive density simplification (green dots). While the EKF-based approach has a constant calculation complexity, in the proposed approach the complexity is reduced over the prediction horizon resulting in an extension of the prediction horizon.

to consider this uncertainty, particularly regarding states in near future. This includes model discrepancies, as well as extrinsic and intrinsic disturbances present in real-world processes. Furthermore, state estimation based on noisy sensor measurements, leads to uncertain state information.

However, explicitly dealing with uncertainties typically leads to a significant increase in computational demand, which generally results in a shortening of the prediction horizon. Apart from that, early knowledge on upcoming states ensures a high similarity between the actual closed-loop input and trajectory, and the predicted open-loop trajectory [11]. Furthermore, it helps to avoid local minima existing in the value function. A long prediction horizon is therefore strongly desirable.

We consider a system with a continuous-valued state space and a finite set of control inputs, where the system states are not directly accessible, i.e., only imperfect state information is available. For example, a mobile robot [12], whose state space is composed of its posture, but only a finite set of movements, as turn left or right and move straight, can be handled. Another example is a sensor network [13] with continuous-valued quantities (i.e., measurement values and related positions) and certain measurement requests as input. In addition, if continuous-valued control inputs are used, the input can be discretized in a reasonable manner.

In this paper, a discrete-time Nonlinear Model Predictive Control for time-variant stochastic systems with continuous state spaces and a finite set of control inputs is presented. The contradiction of simultaneously considering a long prediction horizon and uncertainties in a nonlinear system is overcome by progressive simplification of the occurring probability densities. The basic idea is to lower the computational cost

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by reducing the resolution, i.e., the approximation quality, of the probability densities progressively over the prediction horizon down to a point estimate and, hence, thereby converting the problem into the deterministic case. We propose to apply a specific parametric representation of the underlying probability densities, the so called deterministic Dirac mixture density, where reduction automatically converts the problem into the deterministic case. This approach is illustrated by Fig. 1. Although the idea of combining cheap deterministic strategies with more expensive stochastic calculations is not unknown in general, e.g., in linear MPC [14], it is newly applied within an NMPC framework.

The remainder of this paper is structured as follows: In the following section, the considered NMPC problem is described and illustrated by an example. The applied probability density representation, the deterministic Dirac mixture density, is introduced in Section III. In Section IV, prediction of the state estimate and a complexity reduction method is introduced, for the novel efficient NMPC method. Furthermore, the evaluation of the value function and a complexity analysis is given. In Section V, the introduced method is evaluated by simulations with comparison to three other NMPC controllers. Finally, the paper closes with conclusions and an outlook to future work.

II. PROBLEM FORMULATION

In this paper, we consider a nonlinear discrete-time system

$$\underline{\mathbf{x}}_{k+1} = \underline{a}_k(\underline{\mathbf{x}}_k, \underline{u}_k, \underline{\mathbf{w}}_k),$$

where the random vector $\underline{\mathbf{x}}_k \in \mathcal{X}$ denotes the continuous-valued system state at time step k . The vector-valued control input \underline{u}_k is chosen from the finite discrete set of control inputs \mathcal{U}_k and $\underline{a}_k(\cdot)$ denotes the nonlinear system function, which is possibly time-variant and affected by disturbances denoted by the random vector $\underline{\mathbf{w}}_k$. The system state $\underline{\mathbf{x}}_k$ is assumed to be not directly accessible, but has to be estimated by means of noise corrupted measurements.

Example System: Miniature Walking Robot

A miniature walking robot is supposed to follow a given path, while minimizing the energy cost, i.e., it has to decide upon an optimal trajectory in order to maximize the covered distance.

The robot is able to superimpose left and right turns onto the forward motion, which can be modeled similar to a two-wheeled differential drive. Hence, the movement is given according to

$$\underline{\mathbf{x}}_{k+1} = \begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{y}_{k+1} \\ \phi_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_k + u_k^v \cdot \cos(\phi_k + u_k^\phi) \\ \mathbf{y}_k + u_k^v \cdot \sin(\phi_k + u_k^\phi) \\ \phi_k + u_k^\phi \end{bmatrix} + \begin{bmatrix} \mathbf{w}_k^x \\ \mathbf{w}_k^y \\ \mathbf{w}_k^\phi \end{bmatrix}, \quad (1)$$

where the system state $\underline{\mathbf{x}}_k = [\mathbf{x}_k, \mathbf{y}_k, \phi_k]^T$ denotes the pose of the robot. The control input $\underline{u}_k = [u_k^v, u_k^\phi]^T$ consists of the step length u_k^v and the rotation angle u_k^ϕ . Here the noise $\underline{\mathbf{w}}_k = [\mathbf{w}_k^x, \mathbf{w}_k^y, \mathbf{w}_k^\phi]^T$ affects the system in an additive manner.

Furthermore, the robot is equipped with a sensor for measuring the distance to landmarks according to

$$z_k = \sqrt{(\mathbf{x}_k - \tilde{x}_k)^2 + (\mathbf{y}_k - \tilde{y}_k)^2} + v_k, \quad (2)$$

where $[\tilde{x}, \tilde{y}]^T$ denotes the position of a landmark and v_k subsumes the disturbances affecting the measurement.

At any time step k , the applied controller solves an open-loop feedback control problem over a H step prediction horizon, i.e., the controller has to find an optimal control input sequenz $\underline{u}_{k:k+H-1}^*$ that maximizes the expected value function over the time steps from k to $k+H-1$ based on the current estimated system state $\underline{\mathbf{x}}_k$ incorporating all available information. More precisely, in order to determine the current optimal control input \underline{u}_k^* , the controller has to find the optimal sequenz $\underline{u}_{k:k+H-1}^*$ that maximizes

$$E \left\{ g_{k+H}(\underline{\mathbf{x}}_{k+H}) + \sum_{i=k}^{k+H-1} g_i(\underline{\mathbf{x}}_i, \underline{u}_i) \right\},$$

where $g_k(\cdot)$ denotes the application-specific value function that maps every $\underline{\mathbf{x}}_k \in \mathcal{X}$ and the corresponding $\underline{u}_k \in \mathcal{U}_k$ to a real number. The value function models the desired behavior of the system by rewarding preferred system states and control inputs, and by penalizing undesired state-input pairs. The obtained optimal control input \underline{u}_k^* is then applied to the system and the procedure is repeated in the following time step $k+1$.

State propagation of $\underline{\mathbf{x}}_k, \dots, \underline{\mathbf{x}}_{k+H-1}$ is performed in stochastic systems by means of the so-called Chapman-Kolmogorov equation [15]

$$f_{k+1}^x(\underline{\mathbf{x}}_{k+1}) = \int f_{\underline{u}_k}^T(\underline{\mathbf{x}}_{k+1}|\underline{\mathbf{x}}_k) f_k^x(\underline{\mathbf{x}}_k) d\underline{\mathbf{x}}_k, \quad (3)$$

where $f_k^x(\underline{\mathbf{x}}_k)$ denotes the state density characterizing the system state $\underline{\mathbf{x}}_k$. The transition density $f_{\underline{u}_k}^T(\cdot|\cdot)$ is defined by

$$f_{\underline{u}_k}^T(\underline{\mathbf{x}}_{k+1}|\underline{\mathbf{x}}_k) = \int \delta(\underline{\mathbf{x}}_{k+1} - \underline{a}_k(\underline{\mathbf{x}}_k, \underline{u}_k, \underline{\mathbf{w}}_k)) \cdot f_k^w(\underline{\mathbf{w}}_k) d\underline{\mathbf{w}}_k,$$

where $f_k^w(\underline{\mathbf{w}}_k)$ denotes the noise density characterizing the random vector $\underline{\mathbf{w}}_k$.

In general, the evaluation of (3) cannot be performed in closed form. Approximate solutions to overcome the problem of extrapolating the system states $\underline{\mathbf{x}}_k, \dots, \underline{\mathbf{x}}_{k+H-1}$ have a high complexity, which typically results in a short prediction horizon H .

III. DENSITY REPRESENTATION

Avoiding the high complexity of a complete stochastic extrapolation of system states over the prediction horizon, we propose to reduce the complexity of the occurring probability densities progressively over time at the expense of an increasing approximation error. Utilizing Dirac mixture densities enables an efficient evaluation of arbitrary value functions. Approximation quality and complexity can be controlled by the amount of Dirac components. Furthermore, progressive component reduction naturally results in a point estimate. Hence, at the later timesteps of the prediction horizon, the stochastic NMPC problem is gradually converted into a deterministic NMPC problem, in order to retain a sufficiently long prediction horizon.

Definition 1 (Dirac Mixture Density) An arbitrary N -dimensional random vector $\underline{\mathbf{x}} \in \mathbb{R}^N$ characterized by the

probability density function $f(\underline{x}) : \mathbb{R}^N \rightarrow \mathbb{R}_+$ can be approximated by the Dirac mixture density

$$\tilde{f}(\underline{x}) = \sum_{j=1}^L \omega^{(j)} \cdot \delta(\underline{x} - \underline{x}^{(j)}), \quad (4)$$

where $\underline{x}^{(j)}$, for $j = 1, \dots, L$, denotes the position of the j th Dirac component with the corresponding weight $\omega^{(j)}$ subject to

$$\sum_{j=1}^L \omega^{(j)} = 1.$$

In order to quantify the approximation quality of a probability density, a distance measure has to be defined. Since multidimensionality and comparison between discrete and continuous density representations are difficult to treat, we generalize the relevant distributions by applying the so-called Localized Cumulative Distribution (LCD)

Definition 2 (Localized Cumulative Distribution [16])

For a probability density function $f(\underline{x}) : \mathbb{R}^N \rightarrow \mathbb{R}_+$ characterizing the random vector \underline{x} , the LCD is defined by

$$F(\underline{m}, b) = \int_{\mathbb{R}^N} f(\underline{x}) \cdot K(\underline{x} - \underline{m}, b) d\underline{x},$$

with $F(\cdot, \cdot) : \Omega \rightarrow [0, 1]$, $\Omega \subset \mathbb{R}^N \times \mathbb{R}_+^N$ and $K(\cdot, \cdot) : \Omega \rightarrow [0, 1]$, a symmetric and integrable kernel at position \underline{m} and size b , which is considered to be equal in every dimension.

In the following, we will use a Gaussian type kernel defined by

$$K(\underline{x} - \underline{m}, b) = \prod_{k=1}^N \exp\left(-\frac{1}{2} \frac{(x^{(k)} - m^{(k)})^2}{b^2}\right), \quad (5)$$

where k indicates the dimension. Based on the LCD, the Cramér-von Mises Distance [16] can be modified, resulting in a unique and symmetric distance, which allows the comparison of arbitrary multivariate probability distributions.

Definition 3 (Modified Cramér-von Mises Distance [16])

Given the probability densities $f_1(\underline{x}) : \mathbb{R}^N \rightarrow \mathbb{R}_+$ and $f_2(\underline{x}) : \mathbb{R}^N \rightarrow \mathbb{R}_+$ and their corresponding LCDs $F_1(\underline{m}, b)$ and $F_2(\underline{m}, b)$, the distance D is defined by

$$D = \int_{\mathbb{R}^N} \omega(b) \int_{\mathbb{R}^N} (F_1(\underline{m}, b) - F_2(\underline{m}, b))^2 d\underline{m} db,$$

where $\omega(b) : \mathbb{R}_+ \rightarrow [0, 1]$ denotes a weighting function.

The weighted integral of the squared difference of the LCDs F_1 and F_2 that defines the Modified Cramér-von Mises Distance, intuitively, can be interpreted as comparison of the probability masses in the region of the kernels of the densities f_1 and f_2 . Instead of using a fixed kernel size and position, this distance measure compares the masses for every possible kernel size and position.

Using this distance as the optimization criterion gives rise to a deterministic discrete approximation of arbitrary probability densities. The Dirac positions of the approximation are

optimized by means of minimizing D . For an approximation of Gaussian densities we refer to [17]. Further information about this density approximation method can be found in [16].

IV. STOCHASTIC-DETERMINISTIC NMPC

BASED ON DETERMINISTIC DIRAC MIXTURE DENSITIES

In the previous section, a density representation was introduced, where complexity and approximation quality can be controlled by means of the amount of Dirac components. In this section, the deterministic Dirac mixture density representation is applied to the open-loop control framework by means of prediction including the introduced progressive reduction of the resolution.

A. Prediction

In stochastic NMPC, the current state estimate at time step k is given as a random vector \underline{x}_k , characterized by the probability density function $f_k^x(\underline{x}_k)$. The state estimate \underline{x}_k needs to be predicted over an H step horizon for all inputs $\underline{u}_k \in \mathcal{U}$. For $f_k^x(\underline{x}_k)$ and the noise density $f_k^w(\underline{u}_k)$ approximated by the Dirac mixture densities $\tilde{f}_k^x(\underline{x}_k)$ and $\tilde{f}_k^w(\underline{u}_k)$, respectively, the Chapman-Kolmogorov equation (3) can be solved analytically.

The joint probability density of two Dirac mixture densities can be written as $\tilde{f}_k^z(\underline{z}_k) = \tilde{f}_k^x(\underline{x}_k) \times \tilde{f}_k^w(\underline{u}_k)$, i.e., the Cartesian product of their Dirac components. This leads to the analytical solution of (3) for Dirac mixture densities, which is given by

$$\tilde{f}_{k+1}^p(\underline{x}_{k+1}) = \sum_{i=1}^{L_x \cdot L_w} \omega^{(i)} \cdot \delta\left(\underline{x} - a(\underline{z}_k^{(i)}, \underline{u}_k)\right),$$

where L_x and L_w denote the number of samples for $\tilde{f}_k^x(\underline{x}_k)$ and $\tilde{f}_k^w(\underline{u}_k)$. The predicted density \tilde{f}_{k+1}^p is a Dirac mixture density with $L_x \cdot L_w$ components. Reduction of complexity is realized by approximation of \tilde{f}_{k+1}^p by a new density $\tilde{f}_{k+1}^{\tilde{p}}$. In order to obtain progressive reduction, the amount of Diracs $L_{\tilde{p}}$ of $\tilde{f}_{k+1}^{\tilde{p}}$ is subject to $L_{\tilde{p}} \leq L_x$.

B. Reduction of Dirac Components

An approximation of a Dirac mixture density by another Dirac mixture density can be calculated analytically. As described above, this is done by minimizing the modified Cramér-von Mises Distance with respect to the Dirac positions. Since this distance is based on the introduced LCD, we define the LCD of a Dirac mixture density by applying (5) on (4), which results in

$$\tilde{F}(\underline{m}, b) = \sum_{j=1}^L \omega_j \prod_{k=1}^N \exp\left(-\frac{1}{2} \cdot \frac{(x_j^{(k)} - m^{(k)})^2}{b^2}\right).$$

Given the LCD $\tilde{F}(\underline{m}_y, b)$ of the N -dimensional Dirac mixture density $\tilde{f}^y(\underline{y})$, the LCD $\tilde{F}(\underline{m}_x, b)$ of its approximation $\tilde{f}^x(\underline{x})$, and the weighting function

$$\omega(b) = \begin{cases} \frac{1}{b^{N-1}} & b \in [0, b_{max}], \\ 0 & \text{elsewhere} \end{cases}$$

the modified Cramér-von Mises Distance is defined by

$$D = \int_0^{b_{max}} (D_1 - 2D_2 + D_3) db,$$

with

$$D_1 = \pi^{\frac{N}{2}} b \sum_{i=1}^{L_y} \sum_{j=1}^{L_y} \omega_i \omega_j \exp \left(-\frac{1}{4b^2} \sum_{k=1}^N (y_i^{(k)} - y_j^{(k)})^2 \right),$$

$$D_2 = \pi^{\frac{N}{2}} b \sum_{i=1}^{L_y} \sum_{j=1}^{L_x} \omega_i \omega_j \exp \left(-\frac{1}{4b^2} \sum_{k=1}^N (y_i^{(k)} - x_j^{(k)})^2 \right),$$

$$D_3 = \pi^{\frac{N}{2}} b \sum_{i=1}^{L_x} \sum_{j=1}^{L_x} \omega_i \omega_j \exp \left(-\frac{1}{4b^2} \sum_{k=1}^N (x_i^{(k)} - x_j^{(k)})^2 \right),$$

where i and j denote the Dirac components, ω_i and ω_j the component specific weights, and k indicates the dimension. For large b_{max} the following closed-form expression is given by

$$D_1 = \frac{\pi^{\frac{N}{2}}}{8} \sum_{i=1}^{L_y} \sum_{j=1}^{L_y} \omega_i \omega_j (4b_{max}^2 - C_b T_{ij} + T_{ij} \cdot \log(T_{ij})),$$

with the constant $C_b = \log(4b_{max}^2) - \Gamma$ and $T_{ij} = \sum_{k=1}^N (y_i^{(k)} - y_j^{(k)})^2$. $\Gamma \approx 0.5772$ denotes the Euler gamma constant. Closed-form expressions of D_2 and D_3 are obtained analogously.

In order to obtain the gradient of D , the partial derivative of has to be determined with respect to positions of the Dirac components \underline{x}_ξ of $\tilde{f}^x(\underline{x})$. Since D_1 is constant in terms of \underline{x}_ξ , only D_2 and D_3 have to be differentiated. Hence, the gradient is given by

$$G_\xi^{(\eta)} = \frac{\partial D}{\partial x_\xi^{(\eta)}} = \int_0^{b_{max}} b \pi^{\frac{N}{2}} \left(\frac{\partial D_3}{\partial x_\xi^{(\eta)}} - 2 \frac{\partial D_2}{\partial x_\xi^{(\eta)}} \right) db$$

A closed-form expression can be derived for large b_{max} , which is given by

$$G_\xi^{(\eta)} = \left\{ \frac{\pi^{\frac{N}{2}}}{2} \omega_\xi \left(\sum_{i=1}^{L_x} \omega_i (x_\xi^{(\eta)} - x_i^{(\eta)}) \log \left(\sum_{k=1}^N (x_\xi^{(k)} - x_i^{(k)})^2 \right) + C_b \sum_{i=1}^{L_x} \omega_i (x_\xi^{(\eta)} - x_i^{(\eta)}) \right) \right. \\ \left. - 2 \cdot \left\{ \frac{\pi^{\frac{N}{2}}}{2} \omega_\xi \left(\sum_{i=1}^{L_y} \omega_i (x_\xi^{(\eta)} - y_i^{(\eta)}) \log \left(\sum_{k=1}^N (x_\xi^{(k)} - y_i^{(k)})^2 \right) + C_b \sum_{i=1}^{L_y} \omega_i (x_\xi^{(\eta)} - y_i^{(\eta)}) \right) \right\} \right\},$$

for the component index $\xi = 1, \dots, L_x$ and the dimension index $\eta = 1, \dots, N$.

Moving towards H within the prediction horizon results in an exponential increase of predicted states. Hence, a decreased amount of Dirac components should be used in a suitable way until $L_{\bar{p}} = 1$ is reached and henceforth $\underline{x}_{k+i} = \hat{\underline{x}}_{k+i}$ for $i < H$ holds. Subsequent prediction steps are treated in a deterministic manner.

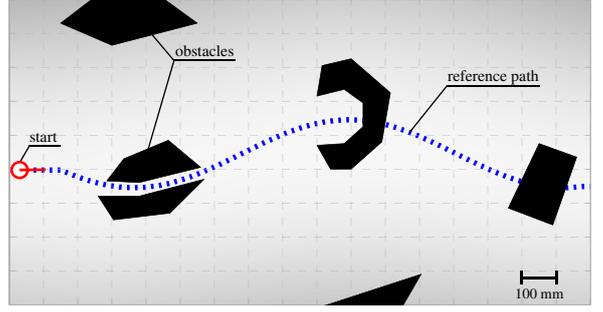


Fig. 2. The used simulation scenario. A mobile robot has to follow a reference path in order to cover a long distance, i.e., decide on an optimal trajectory that minimizes the motion cost. The reference path, however, is obstructed with obstacles. The starting position is illustrated as red icon, the reference path is depicted as blue dotted line, and obstacles are shown as black polygons. The light grey gradient indicates the position cost, subject to the distance to the reference path.

C. Calculation of the Expected Value

Given the current state estimate \underline{x}_k characterized by the probability density $f_k^x(\underline{x}_k)$, the expected value is defined by

$$E \{g(\underline{x}_k, \underline{u}_k)\} = \int g(\underline{x}_k, \underline{u}_k) \cdot f_k^x(\underline{x}_k) d\underline{x}_k. \quad (6)$$

For $f_k^x(\underline{x}_k)$ approximated by the Dirac mixture $\tilde{f}_k^x(\underline{x}_k)$, the expected value can be expressed as

$$E \{g(\underline{x}_k, \underline{u}_k)\} = \sum_{j=1}^L \omega^{(j)} g_k(\underline{x}_k^{(j)}, \underline{u}_k),$$

i.e., the weighted sum of individually evaluated component positions $\underline{x}_k^{(j)}$.

D. Complexity Analysis

Since the system state is being predicted over the entire H step horizon for all $|\mathcal{U}|$ inputs, a brute force realization, i.e., calculation of the entire search tree, leads to exponential complexity $\mathcal{O}(|\mathcal{U}|^H)$. Therefore, we emphasize to use a branch-and-bound algorithm. For an efficient depth first search implementation in the context of MPC dealing with discrete control inputs, we refer to [18].

The introduced stochastic prediction including component reduction can be calculated in $\mathcal{O}(N \cdot L_{\bar{p}} \cdot L_x \cdot L_w)$. For predictions with a large number of Diracs, we refer the reader to [19], where a method is introduced that avoids expansion of the joint probability and, hence, decreases the calculation time to $\mathcal{O}(q \cdot N \cdot L_{\bar{p}} \cdot (L_x + L_w))$. Unfortunately this method requires numerical integration using q integration points. When assuming small L_x , L_w , and thus, small $L_{\bar{p}}$, the introduced analytical method is more efficient.

V. SIMULATIONS

In order to illustrate the advantages of using the presented stochastic NMPC based on progressive density simplification (PDS-SNMPC), we implemented the introduced algorithms based on deterministic Dirac mixture densities and conducted several simulations. The considered simulation scenario is illustrated in Fig. 2, where a miniature walking robot has

TABLE I
SIMULATION RESULTS FOR 100 MONTE-CARLO RUNS.

Method	ϕ distance	collisions	ϕ relative calc. time
PDS-SNMPC (prop.)	1345 mm	0	1.00
det. NMPC	753 mm	85	0.48
EKF based NMPC	1163 mm	9	1.20
DM based NMPC	1169 mm	10	0.93

to follow a given reference path, which is obstructed by obstacles. The system is given by the example system (1) and (2) with a set of the three control inputs

$$\mathcal{U} = \left\{ \begin{bmatrix} 60 \text{ mm} \\ 0 \end{bmatrix}, \begin{bmatrix} 60 \text{ mm} \\ \pm 25^\circ \end{bmatrix} \right\}.$$

We assume a zero-mean Gaussian white noise for $\underline{w}_k = [w_k^x, w_k^y, w_k^\phi]^T$ and \underline{v}_k . The corresponding standard deviations are given by $\sigma_k^x = \sigma_k^y = 3.5 \text{ mm}$, $\sigma^\phi = 1^\circ$, and $\sigma^v = 5 \text{ mm}$, respectively.

The value function is defined by

$$g_k(\underline{x}_k, \underline{u}_k) = \begin{cases} -4 & \text{if true("} \underline{x}_k \text{ in obstacle")} \\ C_m + C_d + C_\alpha & \text{otherwise,} \end{cases}$$

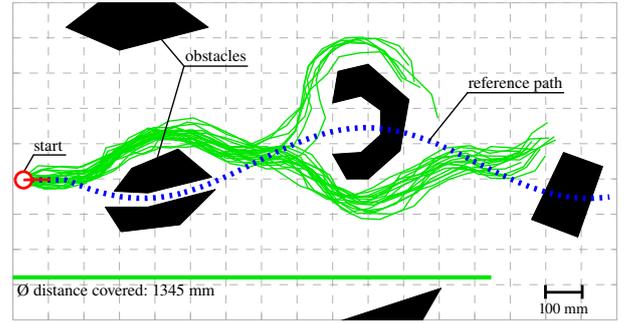
where $C_m = -0.1$ denotes the fixed motion cost. The distance cost is given by $C_d = -0.1 \cdot D(\underline{x}_k)/200$, where $D(\cdot)$ denotes the distance between robot and reference path. Hence, the one-step cost doubles if the robot diverges from the reference path for more than 200 mm. Finally, $C_\alpha = -0.1 \cdot \alpha^2/\pi$ defines the orientation cost. The simulation stops whenever the cumulative cost exceeds $C_{max} = -4$, i.e., 40 steps cannot be exceeded and a collision terminates the run.

The assumed scenario contains two major difficulties. One subsection of the reference path is closely surrounded by obstacles and since the system is noise-corrupted, there is a high probability to hit an obstacle while directly following this path. Another subsection is obstructed with a dead end, which has to be avoided.

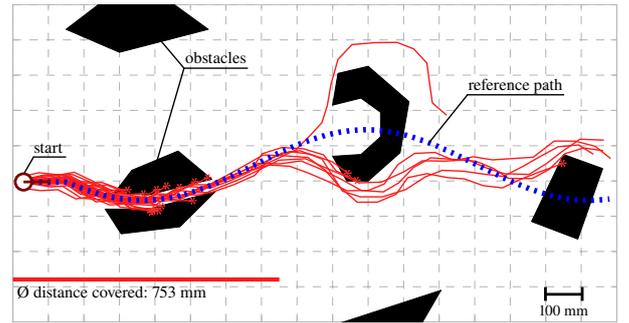
For comparison, a deterministic NMPC method was implemented, as well as two stochastic NMPC controller—namely, a controller based on the Extended Kalman Filter and a controller based on deterministic Dirac mixture densities with a constant number of 20 Dirac components. Evaluation for Gaussian densities was performed by Monte-Carlo sampling with 100 randomly drawn particles.

In order to preserve fair comparison, state estimation was performed by a particle filter with 500 particles and sequential importance resampling [20] in the case of the NMPC methods based on Dirac mixtures and in case of the deterministic NMPC. The state estimate \underline{x}_k is transformed in every time step k into the specific representation, i.e., a deterministic Dirac Mixture and the expected value $E\{\underline{x}_k\}$, respectively. The EKF-based approach uses the Extended Kalman Filter for estimation and, hence, \underline{x}_k is applied directly by the controller.

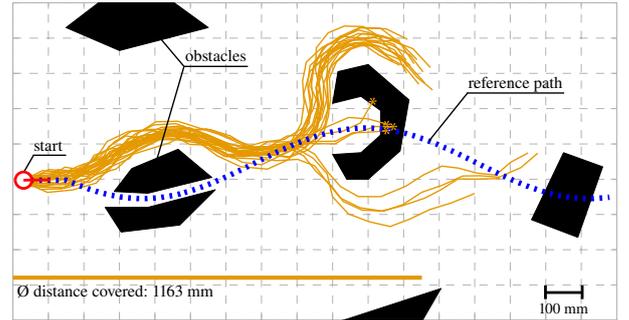
The proposed method was used in a 7 step horizon setup with $L_x = \{25, 20, 14, 8, 1, 1, 1\}$ Dirac components. The deterministic NMPC was used with an 8 step horizon. Both,



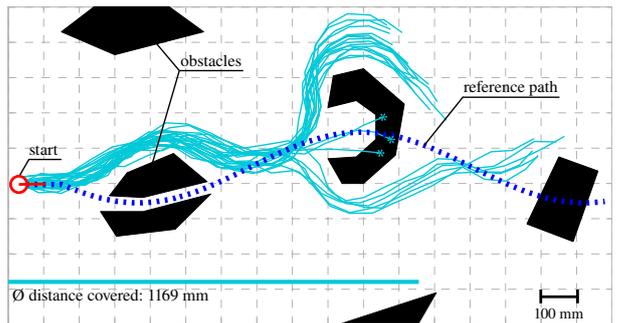
(a) SNMPC based on progressive density simplification (proposed).



(b) Deterministic NMPC.



(c) NMPC based on the Extended Kalman Filter.



(d) NMPC based on Deterministic Dirac Mixture Densities.

Fig. 3. The resulting trajectories of 25 Monte Carlo simulation runs. Subfigure (a) shows the simulation results of the introduced stochastic NMPC based on progressive density simplification, in Subfigure (b) the trajectory of the deterministic NMPC is depicted, Subfigure (c) shows the outcome of the NMPC method based on the Extended Kalman Filter, and in Subfigure (d) the results of the NMPC controller based on deterministic Dirac mixture densities with a constant number of components. The proposed approach perfectly omits the narrow path and the dead end. Advantageous bypasses are found and hence, a great distance is covered. In comparison the deterministic NMPC is not able to avoid the narrow path. Furthermore, other collisions with arbitrary obstacles cannot be prevented. The two remaining stochastic NMPC methods are not able to detect the dead end early in advance, due to the short horizon. Thus, a less favorable bypass is taken and sometimes a collision cannot be avoided.

the EKF-based controller and the deterministic Dirac mixture approach with a constant component number, were utilized with the short horizon of 4 time steps, which is caused by the expensive calculation. While the first suffers from an expensive evaluation of the value function, the second has to deal with a costly prediction.

We conducted 100 Monte-Carlo simulations for each of the compared methods. The results are summarized in Table I. The trajectories of 25 runs are shown in Fig. 3. As can be easily seen, the deterministic NMPC never avoids the narrow path and hence, often hits an obstacle. However, even if the first obstacle is passed the robot is sometimes led into an obstacle, due to the ignorance of disturbances affecting the system. A similar behavior can be seen for both stochastic NMPC controller that use a short prediction horizon. They perfectly avoid the narrow path, but as the short horizon is not fully able to detect the dead end in time, the robot is not always able to avoid all obstacles. Furthermore, due to the short horizon, the second obstacle is mostly bypassed by an expensive route, as the controller can not anticipate the turn of the reference path. In comparison, the proposed approach avoids the narrow path because of the stochastic prediction, but is also able to detect the dead end early. Knowledge about distant areas leads to avoidance of the obstacle, mostly by the cheap bypass route.

VI. CONCLUSIONS

In this paper, a novel approach for open-loop Stochastic Nonlinear Model Predictive Control for time-variant systems with a continuous state space and a finite set of control inputs is presented. We propose to approximate occurring probability densities by deterministic Dirac mixture densities. This parametric density representation enables an efficient evaluation of arbitrary value functions. Moreover, the approximation quality and complexity are naturally controlled by the amount of Dirac components. Stochastic density prediction is progressively transformed over the prediction horizon into deterministic state propagation by means of reduction of the density resolution associated with a shrinking conservativeness. By exploiting the characteristics of deterministic Dirac mixture densities, progressive reduction of Dirac components leads to a point estimate and hence, conversion into the deterministic case is performed automatically. In this way, this approach uses direct incorporation of uncertainties, however, still offers a long prediction horizon. The effectiveness of the presented NMPC approach, as well as the importance of consideration of noise and remote future states are illustrated with simulations of a miniature walking robot following an obstructed path.

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REFERENCES

- [1] D. Bertsekas, "Dynamic Programming and Optimal Control", Athena Scientific, 1995.
- [2] S. J. Qin and T. A. Badgwell, "An Overview of Nonlinear Model Predictive Control Applications", in *Nonlinear Predictive Control*, Springer Basel, pp. 369-392, 2000.
- [3] T. Ohtsuka, "Continuation/GMRES Method for Fast Algorithm of Nonlinear Receding Horizon Control", in *Proceedings of the 39th IEEE Conference on Decision and Control (CDC 2000)*, pp. 766-771, Dec. 2000.
- [4] F. Allgöwer, R. Findeisen, and Z. K. Nagy, "Nonlinear Model Predictive Control: From Theory to Application", in *Journal of the Chinese Institute of Chemical Engineers*, Vol. 35, No. 3, 299-315, 2004.
- [5] J. H. Lee and N. L. Ricker, "Extended Kalman Filter Based Nonlinear Model Predictive Control", in *Industrial & Engineering Chemistry Research*, Vol. 33, pp. 1530-1541, Jun. 1994.
- [6] F. Weissel, M. F. Huber, and U. D. Hanebeck, "A Nonlinear Model Predictive Control Framework Approximating Noise Corrupted Systems with Hybrid Transition Densities" in *Proceedings of the 39th IEEE Conference on Decision and Control (CDC 2007)*, pp. 3661-3666, Dec. 2007.
- [7] F. Weissel, T. Schreiter, M. F. Huber, and U. D. Hanebeck, "Stochastic Model Predictive Control of Time-Variant Nonlinear Systems with Imperfect State Information" in *Proceedings of the 2008 IEEE International Conference on Multisensor Fusion and Integration for Intelligent Systems (MFI 2008)*, pp. 40-46, Aug. 2008.
- [8] F. Weissel, M. F. Huber, and Uwe D. Hanebeck, "Stochastic Nonlinear Model Predictive Control based on Gaussian Mixture Approximations", in *Informatics in Control, Automation and Robotics: Selected Papers from the International Conference on Informatics in Control, Automation and Robotics 2007*, pp. 239-252, Springer, Sept. 2008.
- [9] N. Kantas, J. Maciejowski, and A. Lecchini-Visintini, "Sequential Monte Carlo for Model Predictive Control", in *Nonlinear Model Predictive Control*, Springer Berlin / Heidelberg, LNCIS Vol. 384, pp. 263-273, 2009.
- [10] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A Survey of Recent Results in Networked Control Systems", in *Proceedings of the IEEE*, pp. 138-162, Mar. 2007.
- [11] F. Allgöwer, T. A. Badgwell, J. S. Qin, J. B. Rawlings, and S. J. Wright, "Nonlinear Predictive Control and Moving Horizon Estimation - An Introductory Overview", in *Advances in Control, Highlights of ECC'99*, pages 391-449, Springer, 1999.
- [12] F. Weissel, M. F. Huber, and U. D. Hanebeck, "Test-Environment based on a Team of Miniature Walking Robots for Evaluation of Collaborative Control Methods", in *Proceedings of the 2007 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS 2007)*, pp. 2474-2479, Nov. 2007.
- [13] G. M. Hoffmann and C. J. Tomlin, "Mobile Sensor Network Control Using Mutual Information Methods and Particle Filters", in *IEEE Transactions on Automatic Control*, Vol. 55, No. 1, Jan. 2010.
- [14] L. Blackmore, H. Li, and B. Williams, "A Probabilistic Approach to Optimal Robust Path Planning with Obstacles", in *Proceedings of the 2006 American Control Conference (ACC 2006)*, pp. 2831-2837, Jun. 2006.
- [15] D. Simon, "Optimal State Estimation: Kalman, H Infinity, and Nonlinear Approaches", 1st ed., Wiley & Sons, 2006.
- [16] U. D. Hanebeck and V. Klumpp, "Localized Cumulative Distributions and a Multivariate Generalization of the Cramér-von Mises Distance", in *Proceedings of the 2008 IEEE International Conference on Multisensor Fusion and Integration for Intelligent Systems (MFI 2008)*, pp. 33-39, Aug. 2008.
- [17] U. D. Hanebeck, M. F. Huber, and Vesa Klumpp, "Dirac Mixture Approximation of Multivariate Gaussian Densities", in *Proceedings of the 2009 IEEE Conference on Decision and Control (CDC 2009)*, Dec. 2009.
- [18] J. Wu and S. Abdelwahed, "An Efficient Finite-Input Receding Horizon Control Method and Its Application for the Pneumatic Hopping Robot", in *Journal of Computers*, Vol. 3, No. 9, Sept. 2008.
- [19] H. Eberhardt, V. Klumpp, and U. D. Hanebeck, "Optimal Dirac Approximation by Exploiting Independencies", in *Proceedings of the 2010 American Control Conference (ACC 2010)*, Jun. 2010.
- [20] F. Gustafsson, F. Gunnarsson, N. Bergman, U. Forsell, J. Jansson, R. Karlsson, and P. Nordlund, "Particle Filters for Positioning, Navigation, and Tracking", in *IEEE Transactions on Signal Processing*, Vol. 50, No 2, pp. 425-437, Feb. 2002.