

# Nonlinear Stochastic Model Predictive Control in the Circular Domain

Gerhard Kurz, Maxim Dolgov, and Uwe D. Hanebeck

**Abstract**—In this paper, we present an open-loop Stochastic Model Predictive Control (SMPC) method for discrete-time nonlinear systems whose state is defined on the unit circle. This modeling approach allows considering systems that include periodicity in a more natural way than standard approaches based on linear spaces. The main idea of this work is twofold: (i) we model the quantities of the system, i.e., the state, the measurements, and the noises, directly as circular quantities described by circular probability densities, and (ii) we apply deterministic sampling given in closed form to represent the occurring densities. The latter allows us to make the prediction required for solution of the SMPC problem tractable. We evaluate the proposed control scheme by means of simulations.

## I. INTRODUCTION

In model-based stochastic control, the continuous-valued state and the noises, and thus the corresponding probability densities, are defined in the Euclidean space. However, in some applications, it is more convenient to define these quantities on a manifold such as the circle, e.g., if the system state is given by an angle or another periodic quantity. The definition of the state and the noises in the circular domain allows considering periodicity in a more natural way<sup>1</sup>. Applications where this modeling approach applies are, for example, antenna trackers, radar, robotic joints, rotating cameras, satellite attitude control, FM modulation, and vehicles whose heading is to be controlled [1].

In order to model the stochastic properties of the state and the noises in consistency with their definition space, we apply *circular statistics* [2], [3]. Circular statistics is part of directional statistics [4], a field of statistics that considers probability distributions defined on manifolds representing directional quantities. The motivation for considering directional, or more specifically, circular statistics, is the fact that different probability distributions have to be used on manifolds with periodicity. Whereas the Gaussian distribution arises as a limit distribution for continuous-valued random variables in  $\mathbb{R}^n$ , the so-called wrapped normal distribution arises as a limit distribution on a circle. Traditional measures such as mean, variance, or moments do not apply and, thus, have to be adjusted to the circular setting. To our knowledge, circular statistics has previously been applied only to filtering [5], [6], [7], [8], [9] but not to model-predictive control.

The authors are with the Intelligent Sensor-Actuator-Systems Laboratory (ISAS), Institute for Anthropomatics and Robotics, Karlsruhe Institute of Technology (KIT), Germany. E-Mail: gerhard.kurz@kit.edu, maxim.dolgov@kit.edu, uwe.hanebeck@ieee.org

<sup>1</sup>Please note that systems that are periodic with respect to time constitute a separate research area and are not considered in this paper.

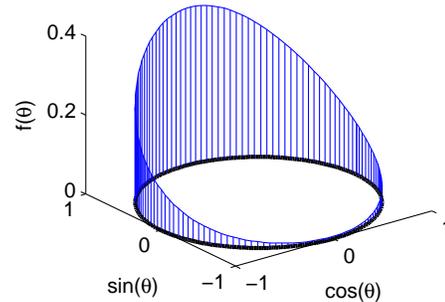


Fig. 1: Probability density function of the wrapped normal distribution with  $\mu = 1.5$  and  $\sigma = 1$ .

Due to periodicity, linear control strategies such as LQG cannot be applied in the circular domain. In this space, a suitable stochastic control method is the Stochastic Model Predictive Control (SMPC). In SMPC, control inputs are obtained by solving online an open-loop control problem at each time step. The control problem consists in finding a control sequence that minimizes a finite-horizon cost function that describes the performance of the controlled system. From the optimal sequence, the first control input is applied to the plant [10]. The advantages of SMPC are, among others, the ability to consider nonlinear systems as well as state and input constraints, the dynamic adaptation of the reference trajectory, etc. [11].

In general, the solution of SMPC problems is not tractable. Therefore, approximations have to be applied. One such class of approximation is *Scenario Approximation*. The key notion of this approach is to represent probability densities of the involved stochastic quantities by a collection of samples, the so called *Dirac mixture*. These samples are then used to compute predictions of stochastic quantities by propagation through the (nonlinear) system function [12], [13], [14].

Control methods that apply scenario approximation can be subdivided according to the method how samples are obtained. *Stochastic sampling* methods use sequential Monte Carlo methods [12]. The main disadvantage of Monte Carlo methods is that due to stochastic sampling they require a large number of samples to provide approximations with sufficiently high confidence. Consequently, computations that apply Monte Carlo methods may become burdensome. In contrast to stochastic sampling, *deterministic sampling* methods place the samples according to a specific rule. The Unscented Transform represents one class of these rules [15]. This approximation method uses  $2n + 1$  samples to represent an  $n$ -dimensional Gaussian distribution. Another class of

deterministic sampling methods places samples such that a specific cost function is minimized. For example, the authors in [13] and [14] employ a method that minimizes a modified version of the Cramér - von Mises distance. The advantage of deterministic sampling is that usually far fewer samples are required to provide a sufficiently good approximation of a probability density. Due to this advantage, the control method presented in this paper relies on deterministic sampling.

The contribution of this work can be summarized as follows. We consider stochastic discrete-time control of periodic systems directly in the circular domain. For this purpose, we assume that the state and the noises are described by wrapped normal probability distributions, which are the natural description of stochastic quantities in the circular domain. To establish tractability of the SMPC's optimization problem, we deterministically approximate the probability densities by means of a Dirac mixture. The approximation is given in closed form.

In the following section, we briefly introduce circular statistics and the wrapped normal distribution. In Sec. III, we formulate the considered problem, and present the proposed solution in Sec. IV. Evaluation results are given in Sec. V and a conclusion in Sec. VI.

## II. CIRCULAR STATISTICS

In Euclidean spaces, the process and measurement noises are often assumed to be Gaussian-distributed because Gaussian probability densities satisfy the central limit theorem. However, if we restrict the system quantities to the unit circle  $S^1$ , where  $S^1$  is parametrized as  $[0, 2\pi)$ , Gaussian densities are no longer suitable. A probability distribution that fulfills a central limit theorem on the circle is the wrapped normal distribution [8], which is obtained by wrapping a normal distribution around the unit circle. The probability density function (pdf) of the wrapped normal distribution is given in the following definition.

### Definition 1 (Wrapped Normal Distribution)

The wrapped normal (WN) distribution  $\mathcal{WN}(\mu, \sigma)$  with parameters  $\mu \in S^1$  and  $\sigma > 0$  is given by

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(x - \mu + 2k\pi)^2}{2\sigma^2}\right).$$

A WN distribution with parameters  $\mu = 1.5$  and  $\sigma = 1$  is depicted in Fig. 1. An important property of the WN distribution is that it is closed under addition of random variables. For random variables  $X \sim \mathcal{WN}(\mu_X, \sigma_X)$  and  $Y \sim \mathcal{WN}(\mu_Y, \sigma_Y)$ , the sum  $Z = X + Y \pmod{2\pi}$  on the circle is distributed according to  $Z \sim \mathcal{WN}(\mu_X + \mu_Y, \sqrt{\sigma_X^2 + \sigma_Y^2})$ . This operation is analogous to convolution of probability density functions in the Euclidean space.

The density approximation method in this paper relies on maintaining the first circular moment of the WN distribution.

### Definition 2 (Circular Moments)

For a circular random variable  $X$  distributed according to a

probability density function  $f(x)$ , the  $n$ -th circular moment is defined as

$$m_n = \mathbb{E}\{\exp(jnX)\} = \int_0^{2\pi} f(x) \exp(jnx) dx,$$

where  $j$  is the imaginary unit.

Since a circular moment is a complex number, each moment has two degrees of freedom. Therefore, the first circular moment contains information about the mean as well as the concentration of the considered probability density. A WN distribution is uniquely defined by its first circular moment.

For a random variable distributed according to  $\mathcal{WN}(\mu, \sigma)$ , the  $n$ -th circular moment can be calculated in closed form according to

$$\begin{aligned} m_n &= \exp\left(jn\mu - \frac{n^2\sigma^2}{2}\right) \\ &= (\cos(n\mu) + j \sin(n\mu)) \cdot \exp\left(-\frac{n^2\sigma^2}{2}\right). \end{aligned} \quad (1)$$

With these prerequisites concerning circular statistics, we are ready to formulate the considered problem in the next section.

## III. PROBLEM FORMULATION

We consider (nonlinear) systems whose state is defined on the unit circle  $S^1$ . We parameterize  $S^1$  as  $[0, 2\pi)$  while keeping in mind the topology of the unit circle. The discrete-time system dynamics are given by

$$x_{k+1} = a_k(x_k, u_k) + w_k \pmod{2\pi}, \quad (2)$$

where  $x_k \in S^1$  denotes the state of the system at time step  $k$ ,  $u_k \in U$  is the control input taking values from a finite<sup>2</sup> set  $U$ , and the independent and identically distributed (i.i.d.) process noise  $w_k \sim \mathcal{WN}(\mu^w, \sigma^w)$  represents endogenous and exogenous uncertainties affecting the system. The time-varying function  $a_k(\cdot) : S^1 \times U \rightarrow S^1$  may be nonlinear.

The state of the system is only available through a measurement process affected by noise according to

$$\hat{z}_k = x_k + v_k,$$

where  $\hat{z}_k \in S^1$  is the measurement and  $v_k \sim \mathcal{WN}(\mu^v, \sigma^v)$  is the independent measurement noise.

In order to quantify system performance over a horizon of the length  $K$ , we use the cumulative cost function

$$J_k^K = \mathbb{E}\left\{g_K(x_K) + \sum_{l=0}^{K-1} g_l(x_{k+l}, u_{k+l})\right\}, \quad (3)$$

where  $g_l(\cdot)$ ,  $l \in \{0, \dots, K\}$ ,  $g_l(\cdot) \geq 0$ ,  $K \in \mathbb{N}$  denotes the stage costs at the corresponding horizon steps. They are defined as

$$\begin{aligned} g_l(x_k, u_k) &= d(x_k, x_k^{ref}) + c(u_k) \text{ and} \\ g_K(x_k) &= d(x_k, x_k^{ref}), \end{aligned} \quad (4)$$

where  $d(x_k, x_k^{ref})$  is the circular distance between the state of the system  $x_k$  and a reference value  $x_k^{ref}$  at time step

<sup>2</sup>An extension to a non-finite input space would be possible using nonlinear optimization techniques.

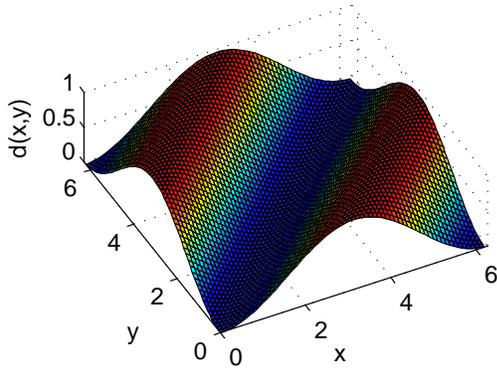


Fig. 2: Angular distance function  $d(x, y)$ .

$k$ , and  $c(\cdot) \geq 0$  is the cost associated with applying control input  $u_k$ . The following definition of  $d(\cdot, \cdot)$  is given in [2].

**Definition 3** (*Angular Distance Function*)

The angular distance function  $d : S^1 \times S^1 \rightarrow [0, 1]$  is defined by

$$d(x, y) = \frac{1 - \cos(x - y)}{2} .$$

The angular distance function (see Fig. 2) is positive definite, symmetric, and invariant to shifting.

To obtain a control input  $u_k$  at time step  $k$ , we seek to solve the finite-horizon open-loop minimization problem defined according to Definition 4. The control input at time step  $k$  is the first element of the optimal sequence  $u_{k:k+K}^*$  that solves the circular SMPC problem. Here, the notation  $a_{k:k+i}$  is an abbreviation for the sequence  $[a_k, a_{k+1}, \dots, a_{k+i}]$ .

**Definition 4** (*Circular SMPC Problem*)

The circular SMPC problem is given by

$$J_k = \min_{u_{k:k+K-1}} \mathbb{E} \left\{ g_K(x_K) + \sum_{l=0}^{K-1} g_l(x_{k+l}, u_{k+l}) \right\} ,$$

s.t.  $x_k \in S^1, u_k \in U$  ,  
 $x_k \sim \mathcal{WN}(\mu^{x_k}, \sigma^{x_k})$  ,  
 $w_k \sim \mathcal{WN}(\mu^{w_k}, \sigma^{w_k})$  ,  
(2), (4) .

The proposed solution to this problem will be presented in the following section. By solving this optimization problem, we obtain the control input  $u_k$  as a function of the current state estimate  $x_k \sim \mathcal{WN}(\mu^{x_k}, \sigma^{x_k})$ .

#### IV. PROPOSED SOLUTION

In this section, we first consider the prediction of the system state in Sec. IV-A and the computation of the expected cumulated costs in Sec. IV-B, both required for the solution of the circular SMPC problem. In Sec. IV-C, we describe how the optimization problem can be solved and how computational issues can be addressed.

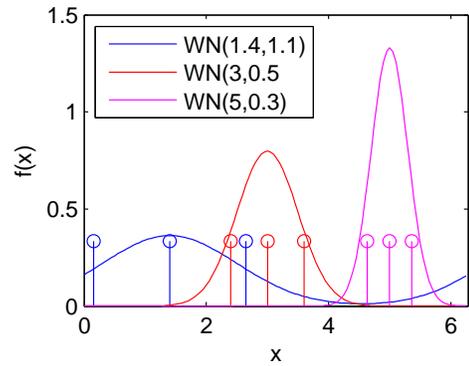


Fig. 3: Three wrapped normal distributions, each approximated with three Dirac components.

#### A. System State Prediction

In order to solve the circular SMPC problem, it is necessary to be able to compute the cumulative costs (3). However, (3) involves the states  $x_{k+l}, l \in \{1, \dots, K-1\}$  predicted based on  $x_k$ . Therefore, the evaluation of (3) requires a propagation of the probability density of the current system state  $x_k$  depending on the potential control inputs. The propagation can be computed according to the Chapman-Kolmogorov equation

$$f(x_{k+1}) = \int f(x_{k+1}|x_k, u_k) f(x_k) dx_k , \quad (5)$$

where  $f(x_k)$  is the (estimated) pdf of the current system state,  $f(x_{k+1}|x_k, u_k)$  is the transition probability density from  $x_k$  to  $x_{k+1}$  in case  $u_k$  is applied, and  $f(x_{k+1})$  is the predicted density of  $x_{k+1}$ . Using the fact that the process noise  $w_k$  is independent from the state  $x_k$ , the transition density  $f(x_{k+1}|x_k, u_k)$  is given by

$$\begin{aligned} f(x_{k+1}|x_k, u_k) &= \int f(x_{k+1}, w_k|x_k, u_k) dw_k \\ &= \int f(x_{k+1}|w_k, x_k, u_k) f^w(w_k) dw_k \\ &= \int \delta(x_{k+1} - a_k(x_k, u_k) - w_k) f^w(w_k) dw_k . \end{aligned} \quad (6)$$

In general, a closed form evaluation of (5) is not possible. We therefore represent the pdf of the state  $x_k$  as a Dirac mixture with  $L$  components

$$\begin{aligned} f(x_k) &\approx \tilde{f}(x_k) \\ &= \sum_{i=1}^L \gamma_k^{(i)} \cdot \delta(x_k - y_k^i) , \end{aligned} \quad (7)$$

where the positions of the samples are denoted by  $y_k^{(i)}$  and their weights by  $\gamma_k^{(i)}$ , with  $\gamma_k^{(i)} > 0$  and  $\sum_{i=1}^L \gamma_k^{(i)} = 1$ .

**Example 1** In this paper, we use a deterministic approximation of  $f(x_k)$  with three components as presented in [8]. This approximation method maintains the first circular moment, which corresponds to the mean and the variance

in the Euclidean domain. For a WN distribution  $\mathcal{WN}(0, \sigma)$  with  $\mu = 0$ , we obtain the Dirac mixture

$$\tilde{f}(x_k) = \sum_{i=1}^3 \gamma^{(i)} \delta(x - y^{(i)}) \quad (8)$$

with positions  $y^{(1)} = -\phi, y^{(2)} = \phi, y^{(3)} = 0$  and weights  $\gamma^{(1)} = \gamma^{(2)} = \gamma^{(3)} = \frac{1}{3}$ , where

$$\phi = \arccos \left( \frac{3}{2} \exp \left( -\frac{\sigma^2}{2} \right) - \frac{1}{2} \right). \quad (9)$$

For WN distributions with  $\mu \neq 0$ , we shift the samples accordingly.

More elaborate deterministic sampling schemes for circular densities have been discussed in [16], [9], and [17].

By representing the density  $f(x_k)$  according to (7) and using (6), the evaluation of (5) yields

$$\begin{aligned} f(x_{k+1}) &= \int f(x_{k+1}|x_k) f(x_k) dx_k \\ &\approx \int f(x_{k+1}|x_k) \sum_{i=1}^L \gamma_k^{(i)} \delta(x_k - y_k^{(i)}) dx_k \\ &= \int \sum_{i=1}^L \gamma_k^{(i)} \int \delta(x_{k+1} - a_k(x_k, u_k) - w_k) \\ &\quad \cdot \delta(x_k - y_k^{(i)}) dx_k f^w(w_k) dw_k \\ &\approx \int f^a(x_{k+1} - w_k) f^w(w_k) dw_k \\ &= (f^a * f^w)(x_{k+1}), \end{aligned}$$

where  $f^a(x)$  is obtained by fitting a WN distribution to the Dirac mixture

$$\sum_{i=1}^L \gamma_k^{(i)} \delta(x - a_k(y_k^{(i)}, u_k))$$

as described in [8], and  $f^a * f^w$  denotes the convolution of both densities (see Sec. II).

### B. Computation of Cumulated Costs

Another challenge when solving the circular SMPC problem is the computation of the expected value in (3). We can rewrite the cost function (3) according to

$$\begin{aligned} J_k &= \mathbb{E} \{g_K(x_K)\} + \sum_{l=0}^{K-1} \mathbb{E} \{g_l(x_{k+l}, u_{k+l})\} \\ &= \mathbb{E} \left\{ d(x_K, x_K^{ref}) \right\} + \sum_{l=0}^{K-1} \left( \mathbb{E} \left\{ d(x_k, x_k^{ref}) \right\} + \mathbb{E} \{c(u_{k+l})\} \right), \end{aligned}$$

with

$$\begin{aligned} \mathbb{E} \left\{ d(x_t, x_t^{ref}) \right\} &= \mathbb{E} \left\{ \frac{1 - \cos(x_t - x_t^{ref})}{2} \right\} \\ &= \frac{1}{2} \left( 1 - \mathbb{E} \left\{ \cos(x_t - x_t^{ref}) \right\} \right). \end{aligned}$$

**Input:** prior density  $f(x_k|y_{1:k})$  as WN distribution

**Output:** control input  $u_k^*$

$J_k^{\min} \leftarrow \infty$ ;

// iterate over all control sequences

**for**  $(u_k, \dots, u_{k+K-1}) \in U^K$  **do**

$J_k \leftarrow 0$ ;

    // iterate over horizon

**for**  $t \leftarrow 0$  **to**  $K - 1$  **do**

        // perform prediction

        Approximate the pdf  $f(x_{k+t})$  by means of a Dirac mixture using (8), (9);

        Propagate the samples through the process equation (2) for input  $u_{k+t}$ ;

        Fit WN to samples;

        Convolve with system noise;

        // calculate expected cost

$J_k \leftarrow J_k + \mathbb{E} \left\{ d(x_{k+t}, x_{k+t}^{ref}) + c(u_{k+t}) \right\}$ ;

**end**

$J_k \leftarrow J_k + \mathbb{E} \left\{ d(x_{k+K}, x_{k+K}^{ref}) \right\}$ ;

    // find optimum

**if**  $J_k < J_k^{\min}$  **then**

$J_k^{\min} \leftarrow J_k$ ;

$u_k^* \leftarrow u_k$ ;

**end**

**end**

**return**  $u_k^*$ ;

**Algorithm 1:** Brute force SMPC algorithm.

For the expected value  $\mathbb{E} \left\{ \cos(x_t - x_t^{ref}) \right\}$ , it holds

$$\begin{aligned} \mathbb{E} \left\{ \cos(x_t - x_t^{ref}) \right\} &= \int_0^{2\pi} \cos(x_t - x_t^{ref}) f(x_t) dx_t \\ &= \int_0^{2\pi} \left( \cos(x_t) \cos(x_t^{ref}) + \sin(x_t) \sin(x_t^{ref}) \right) f(x_t) dx_t \\ &= \cos(x_t^{ref}) \int_0^{2\pi} \cos(x_t) f(x_t) dx_t \\ &\quad + \sin(x_t^{ref}) \int_0^{2\pi} \sin(x_t) f(x_t) dx_t \\ &= \cos(x_t^{ref}) \cos(\mu_t^p) \exp(-(\sigma_t^p)^2/2) \\ &\quad + \sin(x_t^{ref}) \sin(\mu_t^p) \exp(-(\sigma_t^p)^2/2) \\ &= \cos(x_t^{ref} - \mu_t^p) \exp(-(\sigma_t^p)^2/2), \end{aligned}$$

where we apply the trigonometric addition formula

$$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$$

twice. Furthermore, we use the fact that  $\int_0^{2\pi} \cos(x) f(x) dx$  is the real part and  $\int_0^{2\pi} \sin(x) f(x) dx$  is the imaginary part of the first circular moment of a WN distribution as given in (1).

### C. Solution to the Circular SMPC Problem

The optimal solution to the circular SMPC problem stated in Definition 4 can be obtained by computing the costs (3) for all possible sequences  $u_{k:k+K-1}$  and then selecting the sequence that induces minimal costs.

Before solving the circular SMPC problem, the controller has to estimate the pdf  $f(x_k)$  of the current state based on the available measurements. The recursive circular filter presented in [8] can be used to perform this task.

To compute the optimal solution of the circular SMPC problem, it is necessary to construct a search tree for all possible sequences of control inputs and then determine the sequence with the lowest costs. This brute-force version of the control law that is performed by the controller at each time step is summarized in Algorithm 1. However, it leads to complexity  $\mathcal{O}(|U|^K)$ , which is exponential in  $K$ . To reduce the computational burden, we adapt the idea published in [18]. For this purpose, we exploit the fact that the stage costs  $g_l$  can be bounded as follows

$$\bar{g}_l(x_k, u_k) = 1 + \max_{u \in U} c(u) .$$

Thus, at every time step  $l$  of the optimization horizon, we can bound the costs-to-go, i.e., the costs from time step  $l$  to  $K$ , of every leaf of the search tree. This bound can then be used to dismiss leaves whose costs are already too high, i.e., there exists at least one leaf whose worst-case costs of the complete optimization horizon will be smaller than the costs of the considered leaf up to  $l$ .

## V. EVALUATION

The performance of the proposed approach is evaluated with Monte Carlo simulations. We consider a system with the process equation (2)

$$x_{k+1} = a_k(x_k, u_k) + w_k \quad \text{mod } 2\pi ,$$

where  $a_k(x_k, u_k) = x_k + u_k + r \cdot \sin(x_k + u_k)$ . The parameter  $r \in \mathbb{R}$  influences the nonlinearity. We consider two scenarios,  $r = 0$  and  $r = 0.5$ .

The set of control inputs is given by  $U = \{-1, 0, 1\}$  and the associated costs are defined as

$$c(u_k) = \begin{cases} 0, & u_k = 0 \\ 0.1, & u_k \neq 0 \end{cases} .$$

For the purpose of our simulations, we consider a planning horizon of  $K = 3$ . The reference trajectory is given by

$$x_k^{ref} = \pi \cdot \sin(k/20) \quad \text{mod } 2\pi .$$

The process noise and measurement noise have parameters  $\sigma^{w_k} = 0.01$  and  $\sigma^{v_k} = 2$ , i.e., there is a high measurement noise and a low process noise. Furthermore, we assume the initial estimate  $x_0$  to be very poor with the initial uncertainty  $\sigma_0 = 3$ .

For comparison, we implemented a solution to the SMPC problem based on the unscented Kalman filter (UKF) [19]. Both estimation and control use the same basic algorithm but are performed based on Gaussian distributions and the UKF sampling rather than the WN distribution and the presented sampling algorithm.

An example run of the simulation is depicted in Fig. 4. For each time step, the ground truth (i.e., true state of the system), the estimated state of the system, the measurement and the value of the desired trajectory are shown (left). As can be seen, the measurements are extremely noisy, but still,

the estimate converges to the true state of the system and the controller is able to stay close to the desired trajectory by applying the control inputs shown on the right. In order to evaluate the different approaches, we performed 100 Monte Carlo runs and compared the total costs accumulated over the complete run consisting of 200 time steps. The results are depicted in Fig. 5. Lower costs indicate a better result according to the cost function as given in Definition 4. The results show the superiority of the proposed method based on WN distributions for both the linear and the nonlinear system functions. It should be noted that the computational cost of the proposed approach is similar to that of the UKF-based solution.

## VI. CONCLUSION

In this paper, we considered stochastic model-based predictive control in the circular domain. Unlike previous approaches, the proposed method relies on circular probability distributions that originate in the field of directional statistics. By doing so, we are able to consider periodicity in a more natural way. For this setup, we presented a novel stochastic model predictive control method. In order to make the SMPC optimization procedure tractable, we deterministically approximated the occurring probability densities by means of a Dirac mixture.

The presented method has been evaluated in simulations and shown to be superior to the approach based on the UKF. This can be explained by the fact that the proposed method explicitly considers the circular nature of the problem, which is neglected in general approaches.

Future work may include the extension of the proposed methods to higher-dimensional spaces, e.g., MIMO systems, control on the rotation group  $SO(3)$ , etc. This kind of generalization is nontrivial, but there is already some work on the topic of estimation in these scenarios [20], [21], which could be used as a foundation. Furthermore, stability issues could be considered and upper bounds for the error in the approximations could be shown. Also, a closed-loop solution would be an interesting extension to the work presented in this paper.

## ACKNOWLEDGMENT

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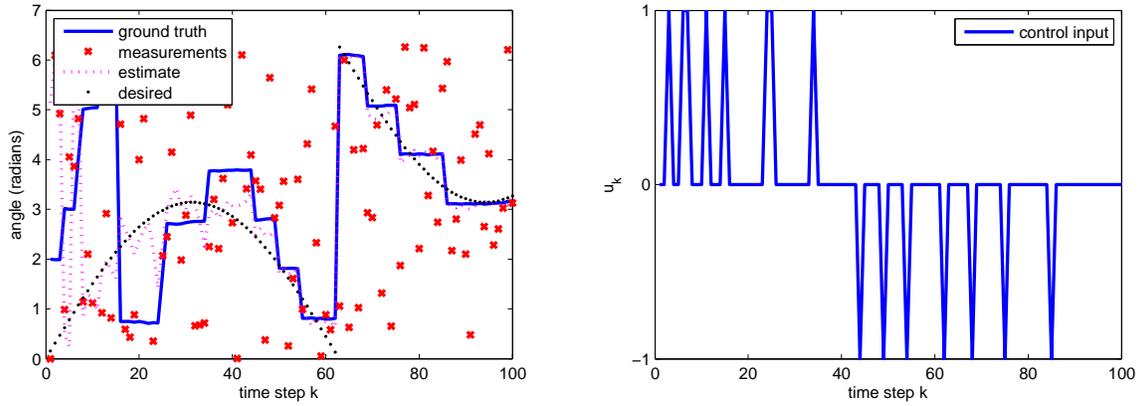


Fig. 4: This figure shows an example run with  $r = 0$ . Keep in mind that the state, i.e., the vertical axis, is  $2\pi$ -periodic.

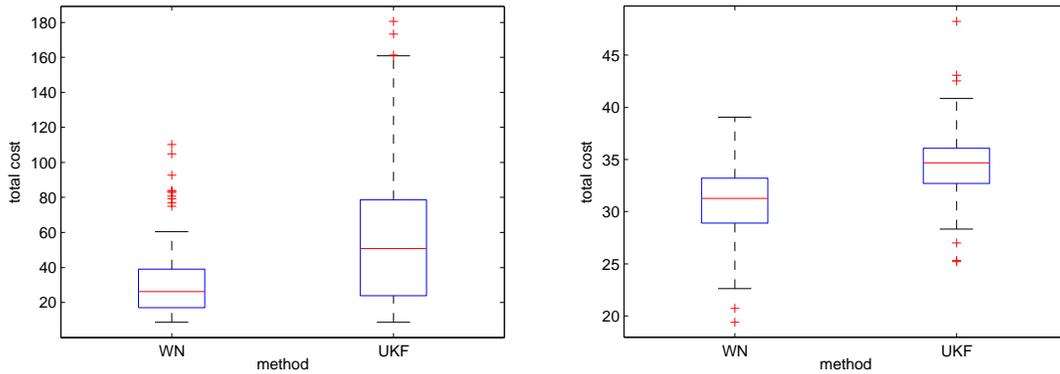


Fig. 5: Evaluation results from 100 Monte Carlo runs for the linear scenario  $r = 0$  (left) and the nonlinear scenario  $r = 0.5$  (right).

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